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M.V. Meerov

MULTIVARIABLE CONTROL SYSTEMS

TRANSLATED FROM RUSSIAN

Published for the National Aeronautics and Space Administration
and the National Science Foundation, Washington, D.C.
by the Israel Program for Scientific Translations

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M.V. Meerov

MULTIVARIABLE CONTROL SYSTEMS

(Sistemy mnogosvyaznogo regulirovaniya)

Izdatel'stvo Nauka
Glavnaya Redaktsiya
Fiziko-Matematicheskoi Literatury
Moskva 1965

Translated from Russian

Israel Program for Scientific Translations
Jerusalem 1968

NASA TT F-435
TT 67-51371
Published Pursuant to an Agreement with
THE NATIONAL AERONAUTICS AND SPACE ADMINISTRATION, U.S.A.
and
THE NATIONAL SCIENCE FOUNDATION, WASHINGTON, D.C.

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Israel Program for Scientific Translations Ltd.
IPST Cat. No. 1820

Translated by Z. Lerman

Printed in Jerusalem by S. Monson
Binding: Wiener Bindery Ltd., Jerusalem

Available from the
U.S. DEPARTMENT OF COMMERCE
Clearinghouse for Federal Scientific and Technical Information
Springfield, Va. 22151

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PREFACE

A characteristic feature of modern industrial and production processes is that their qualitative and quantitative parameters are a function of many interdependent and interconnected variables. Some of the process variables must be maintained constant or made to vary in a manner prescribed by the characteristic features of the given process. These are the so-called controlled variables of the process. Their number is not fixed, and some fairly complex systems may have but a single controlled variable. Such single-variable systems are treated very extensively in the current literature on automatic control theory.

The present book, on the other hand, is devoted to automatic control systems with many controlled variables (at least more than one).

Examples abound of systems with numerous controlled variables, and the modern tendency is toward ever greater utilization of systems and plants of this kind. We call them multivariable control systems (MCS).*

The simplest examples of multivariable plants are provided by complex industrial equipment. Boilers, synchronous electrical machines, etc., are typical examples. In these machines some variables, e.g., steam pressure, steam temperature, voltage, a.c. frequency, are maintained at a certain setting, although the total number of variables (the number of generalized degrees of freedom) is much higher.

The development of multivariable control systems led to a new problem: how to control each of the variables if they are interdependent, so that a change in one of the variables alters all the others? The solution was provided by I.N. Voznesenskii, who can be regarded as the originator of the theory of autonomous, noninteracting control systems: the basic idea was to design a control system with independent variables, where variation of one variable did not change the other variables. This approach proved to be quite useful for a number of controlled objects and it is currently the only practicable solution of the problem in some cases.

However, this solution is inapplicable to most multivariable objects, and in certain cases it is even meaningless. There is a by-now classical illustration of this point. In continuous cold or hot rolling of sheet metal, the controlled variables include the drive speeds, roll gaps, etc., but the quality and mainly the geometry of the finished product do not depend on each controlled variable separately, but on their combination, so that control of each individual variable ignoring all the others at any given time is a meaningless procedure.

Therefore, in addition to controlled objects which technologically can be treated as noninteracting, there are cases of inherently interacting variables, which cannot be adjusted individually. In the latter class we

* A more rigorous definition of a multivariable control system is given in the following.

put all the plants or processes where the generalized quality index of the finished product depends on all the controlled variables simultaneously. It is shown in this book that the design of noninteracting systems is not always the best policy, not even for controlled objects where this is technically feasible. We must emphasize, however, that there are very numerous cases when noninteraction is simply unfeasible.

These two classes, however, are not separated by a Chinese Wall, as everything depends on the problem being considered and the particular conditions. For example, a synchronous electric machine as such falls in the first category, whereas the same machine as part of a power transmission system is an excellent example of a component in a system with inherently interacting variables.

The basic problem of the present book is to elucidate the fundamental properties of multivariable control systems. Whenever possible, I tried to assess and evaluate the current methods and techniques for the synthesis and analysis of control systems and to describe some of my original results.

The book comprises an introduction and eight chapters. The introduction outlines the scope of the treatment and defines the fundamental concepts.

Chapter One is devoted to mathematical description of some typical mutlivariable objects and control systems. The choice of the examples is largely determined by my own field of interest. However, it seems to me that the examples of Chapter One are of general significance as being representative of the principal branches of industry — metallurgy, power engineering, oil engineering and oil refining. The derivation of the equation of the rectifying column and the analysis of its behavior as a control system were carried out by Yu. N. Mikhailov under my supervision.

Chapter Two is devoted to the derivation of the equations of multivariable control systems consisting of single-variable subsystems that are made up of basic (necessary but not sufficient) elements. It will become clear from what follows that this is not a fundamental restriction, since the technique used in the derivation of the equations and the methods employed in their investigation are applicable to the more general cases too. The principal structural properties of this class of systems are elucidated for both the steady-state and transient conditions. In particular, the matrix of error coefficients is determined for the case of plant and control coupling of the individual variables.

Chapters Three and Four investigate the general structural properties of multivariable control systems. The main emphasis is on the class of structures with infinite-gain stability in each subsystem; in these structures every single-variable subsystem is clearly a multiloop configuration.

Multivariable combined-control systems are treated separately in Chapter Five. Considerable space is devoted, in particular, to systems where simultaneous deviation and load control is applied to structures of infinite-gain stability.

Chapter Six deals with the problems of noninteraction and invariance. The presentation begins with a discussion of the results of Voznesenskii (USSR) and of Boksenbom and Hood (USA). We then proceed with the invariance problem and describe the fundamental results of Kulebakin and Petrov. Next, noninteraction and invariance are treated as structural properties of a certain class of systems. Realizability and coarseness (in the sense of A. A. Andronov), various cases of noise rejection, etc., are considered in great detail.

Chapter Seven is concerned with the design of fixed-structure systems which are equivalent in their properties to self-adjusting or adaptive control systems. The discussion is based on structures with infinite-gain stability, which have been treated in considerable detail in the preceding chapters. The structural aspect of the sensitivity problem is dealt with, and examples of systems with variable coefficients are examined. The theoretical results are applied to a practical control problem accommodating a large variation of the plant gain.

Chapter Eight is concerned with the variational aspects of multivariable control. Optimization considerations suggest that the multivariable control systems should be divided into two classes: systems where the general optimum is attained by optimizing each single-variable subsystem, ignoring the interaction with other controlled variables (in this class optimization is synonimous to noninteraction) and systems with a generalized quality index which depends simultaneously on all the controlled variables. A particular example is considered where the control specifications are given as a function of time. Here of particular interest are plants without memory, to which linear programming can be successfully applied. This range of problems was studied jointly by me and E.S. Silimzhanov. The results of Sarachik and Kranc, also discussed in Chapter Eight, are of considerable interest for the determination of the control vector as a function of time for multivariable objects. Classical variational techniques and dynamic programming are applied to determine the controller equations in a multivariable system. It is remarkable that the solution of the variational problem in an open domain yields structures with an infinite gain parameter. On the whole the treatment of this chapter can be regarded as only the first step toward a comprehensive solution of the problem of synthesis of multivariable control systems.

It is my pleasant duty to thank Prof. A.A. Fel'dbaum who reviewed the book and offered a number of highly valuable comments which greatly contributed to the finished product.

M. Meerov

INTRODUCTION

In multivariable objects or plants the number of controlled variables is greater than one and in general these variables are interconnected in such a way that a change in any of the controlled variables alters all the other variables (this refers to steady-state conditions, as well as to transients).

If the controlled variables are regarded as the plant outputs, we may say that a multivariable plant has more than one output, and a change in any one of its outputs leads to a change in all the outputs.

If a closed control loop is hooked up for each of the controlled variables, we end up with a multivariable control system.

Multivariable control systems (MCS) are thus defined in general as control systems with several controlled variables which are coupled in such a way that a change in any one variable leads to a change in all the variables, assuming of course that no special decoupling device is provided.

Typical multivariable objects are a boiler, where the controlled variables are temperature, steam pressure, and water level; a turbojet engine, where both the revolutions and the gas temperature at the turbine outlet are controlled; a synchronous generator, where the voltage and the speed are controlled (if the synchronous generator is connected in parallel with other machines, the active and reactive power output are additional controlled variables).

In the above examples the interrelationship between the individual variables is due to natural (internal) properties of the controlled object. Another extensive group of multivariable control systems arises in connection with automation of production processes. The interaction between the individual controlled variables in these systems is generally due to technical and production factors. An excellent example is the feedback control system for the electric drives in hot and cold continuous rolling mills.

Figure I-1 is a block diagram of a system controlling the sheet thickness in continuous cold rolling. Thickness gages (TG) are provided after each

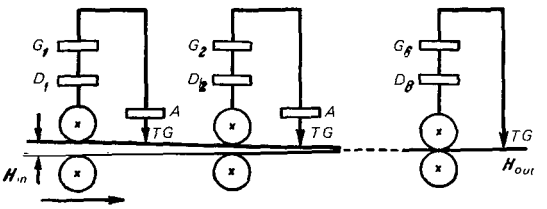

FIGURE I-1. Block diagram of strip gage control in a continuous rolling mill.

stand. The sheet thickness is regulated by adjusting the roll gap and maintaining constant rolling stress. The gage output signal is delivered to the servosystem controlling the pressing screws. The rolling stress, on the other hand, is maintained constant by adjusting the speed ratio of the main drives and the coiler speed. These two groups of control systems, however, are interconnected through the rolled metal strip, and thus constitute a complex multivariable system.

The situation is considerably more complicated in hot rolling. Here the thickness gage can be installed only after the last stand; moreover, it is desirable to control the strip thickness at minimum permissible tension. In hot rolling the strip thickness is highly sensitive to tension. Variation of strip temperature and the heating of rolls also have a considerable influence; there is always a certain contribution from other entirely random factors as well. The object of control is to maintain the strip thickness δ constant. The gage δ depends on the position of the pressing screws, the speed ratio of the main drives, temperature, and other random factors:

$$\delta = f[F_{\text{pr}}(t), F_{ni, ni-1}(t), \Theta, \beta(x_i)]. \qquad \text{(I-1)}$$

Here $F_{\text{pr}}(t)$ is the control function of the pressing screws in the stands, $F_{ni, ni-1}(t)$ is the control function of the main drives, Θ is the temperature, $\beta(x_i)$ is a disturbance dependent on random factors.

We see from (I-1) that the controlled variable depends on the determinate functions $F_{\text{pr}}(t)$, $F_{ni, ni-1}(t)$, and a random function $\beta(x_i)$. The functions $F_{\text{pr}}(t)$ and $F_{ni, ni-1}(t)$ are interrelated, and they jointly determine the geometry and, in particular, the thickness of the rolled strip. The control problem here is to choose the functions $F_{\text{pr}}(t)$ and $F_{ni, ni-1}(t)$ and the function f for given Θ and known probability distribution of $\beta(x_i)$ so that the thickness δ is between predetermined limits.

In rolling mills the strip tension control system and the roll positioning system are coupled through the metal strip. The system for primary refining and sulfur stabilization of crude oil (dehydration and desalination) comes under the same category; the controlled variables here are temperature, flow rate, and liquid level, as well as the quantity of the chemical reagent which is fed separately into the system. The control function should be so chosen that oil of desired quality is obtained at a minimum cost.

There are many other examples from modern industry and technology where the desired quality of the finished product is ensured by simultaneously controlling a number of variables. The controlled variables are generally coupled, so that a change in some of the variables leads to a change in all the variables. We can safely say that the multivariable control theory provides a theoretical foundation to large-scale comprehensive automation of industrial and technological processes.

The third group of multivariable control systems comprises the so-called multidimensional servosystems. These are derived from ordinary servosystems by imposing coupling on the measuring elements. In this case we speak of the coupling of the component servos through the measuring devices or control coupling. Figure I-2 shows a two-dimensional servosystem, and Figure I-3 a three-dimensional servosystem. This combination of individual servos into a single multidimensional

system may be due to the particular requirements of the technological process, e.g., a copying machine. In some cases it also helps to improve the quality of automatic control.

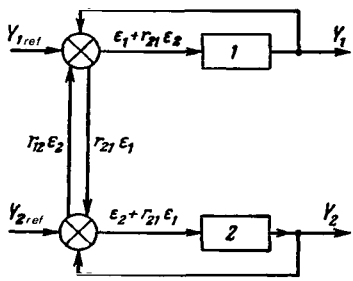

FIGURE I-2. A two-dimensional servosystem.

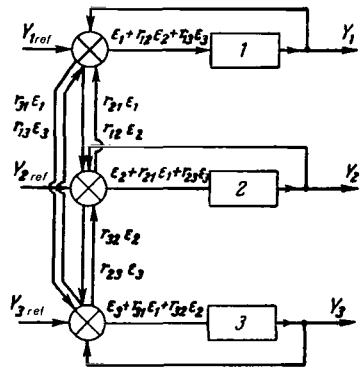

FIGURE I-3. A three-dimensional servosystem.

A common property of all multivariable control systems is that they have several controlled variables (more than one). A separate subsystem is designed for each controlled variable. The number of controllers is naturally at least equal to the number of controlled variables. In multivariable plants the number of inputs is not less than the number of outputs. It will be clear from the following examples that the number of controllers (active inputs) is often greater than the number of controlled variables. Moreover, the controlled object is subjected to external disturbances which may vary arbitrarily (and are often described by random functions). External disturbances, or loads, can be applied to some of the controlled variables or to all of them. A multivariable control system thus contains all the component elements which are normally encountered in systems with one controlled variable.

However, the presence of several controlled variables constitutes more than a simple quantitative difference between multivariable and single-variable systems. There are some special problems which are characteristic of multivariable control systems, and it would be incorrect to assume that the multivariable control theory is a simple generalization of the control theory for systems with a single variable.

For example, let us consider the problem of constraints imposed on the system. In single-variable systems these constraints are mainly determined by the nonlinearity of the characteristics, saturation phenomena, etc., whereas in multivariable systems the constraints may be connected with the peculiar character of the controlled variables. The presence of several coupled controlled variables is a novel aspect in stability analysis and quality considerations, not encountered in single-variable systems. The study of multivariable systems also gives rise to certain topics without counterpart in conventional control theory, such as (a) the problem of noninteraction, (b) the problem of maintaining a given relationship

between the controlled variables, and (c) the problem of interacting control, which minimizes (or maximizes) a certain quantity (e.g., the quality of the finished product in a technological process).

Structure synthesis, which has emerged as one of the basic problems in single-variable control systems, acquires special significance in multivariable control. It will be seen from the various sections of this book that the coupling between the individual controlled variables essentially depends on the structure of the multivariable system, and noninteraction may be derived as a structural property of a certain class of structures. The contribution from inherently nonlinear characteristics of the various elements and their influence on the coupling between the variables and, in particular, on the noninteraction aspects deserve special consideration. Finally, the optimum and extremum problems are of special interest for multivariable control systems. It will be shown that for a certain class of structures noninteraction is equivalent to optimizing the system with respect to some quality criterion.

The realizability of the invariance conditions also has some unique aspects for multivariable control systems. The invariance conditions of multivariable control are realizable only in combined-control systems, where control by deviation (the Watt—Polzunov principle) is implemented in conjunction with load control. Fairly extensive space is allotted in this book to the treatment of combined-action control systems.

We have already noted that the quality of a multivariable control system is often determined by a generalized criterion. The control functions for each variable should be so chosen as to extremize the generalized quality index. In some cases, linear programming provides an effective tool for the development of multivariable control systems of this kind. In Chapter Eight linear programming is applied to find the optimum operating conditions of oil wells. We seek to maximize the oil production under given constraints on equipment and operating conditions. Some economic index (e.g., production costs) can be adopted as the generalized criterion in this case.

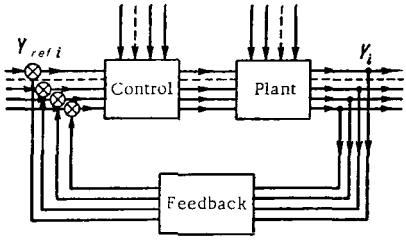

FIGURE I-4. A generalized block diagram of a multivariable system.

The multivariable control theory is very intimately linked with the problem of efficient design of large systems. However complex the system, it always has a certain finite number of main outputs, although there may be any number of factors actively influencing these outputs. Moreover, the statistical indices of the process may be adopted as the generalized outputs. The significant point is that even in these complex systems we can always detect the main outputs, which are interconnected in a certain way and acted upon by additional random disturbances. On the whole, a complex multivariable control system can be represented by some generalized block diagram, like the one shown in Figure I-4.

And now some history. The first serious contributions in multivariable control theory were published in the Soviet Union in 1938 /10, 11/. These initial efforts were entirely concerned with the problem of noninteraction. Voznesenskii /11/ considered the feasibility of providing separate controllers for the individual variables and setting up such coupling that a change in one of the variables would not affect the other variables. This noninteraction problem, which Voznesenskii called the problem of autonomous control, is solved in /11/ for the case of a plant where each variable is described by a first-order differential equation; ideal (inertialess) controllers are assumed. The followers of Voznesenskii have extended the noninteraction conditions to more complex cases /5, 6, 18, 52, 54, 55, 59/. It should be emphasized that the noninteraction problem is figured as the main topic in most studies in the field.

Boksenbom and Hood were the pioneers of multivariable control theory in the USA. Their first paper /77/ published in 1950 deals with various aspects of the noninteraction problem, previously treated by Voznesenskii. The application of matrix algebra enabled the authors to essentially simplify the expressions for noninteraction conditions, without restricting the order of the differential equation that describes each of the controlled variable. The studies of Freeman /78, 79/, Kavanagh /81, 82/, and others concerned with more elaborate aspects of noninteracting systems were a direct outgrowth of the fundamental study of Boksenbom and Hood. Kavanagh considered not only noninteraction, but also some other quality indices. Golomb and Usdin /80/ developed the theory of multivariable servosystems; they introduced the matrix of error coefficients and derived an explicit expression of this matrix for multidimensional servosystems.

Sarachik /21/ considered some properties of nonlinear multidimensional servosystems. He analyzed in considerable detail the properties of a two-dimensional servosystem and described methods of construction that satisfied his optimality test. Multivariable control is also the subject of /15, 83, 84, 85/. The book by M. Mesarović deserves special mention /85/. This was essentially the first book in multivariable control theory; moreover, Mesarović was the first to consider multivariable control as an independent problem, and not as an outgrowth of the theory of single-variable systems. He advanced a number of highly original ideas concerning the applicability of variational techniques to the design of multivariable systems.

Among the more recent contributions to multivariable control theory we should mention the publications of A.A. Krasovskii /22, 23, 24/, V.T. Morozovskii /48, 49, 50/, V.A. Venikov /9/, L.V. Tsukernik /69, 70/, G.V. Mikhnevich /46, 47/, and others. Numerous papers on multivariable control systems have been lately stimulated by research in nuclear-reactor control /12, 45/. On the whole, however, the multivariable control theory is still at the very first stages of its development.

In writing this book I did not try to cover the entire range of problems treated in multivariable control systems. My principal aim was to provide the reader with an introduction to the modern tasks and problems of multivariable control theory and to draw the attention of the specialist to some of the important problems that deserve further study.

Chapter One

EXAMPLES OF MULTIVARIABLE CONTROL SYSTEMS

§ 1.1. AUTOMATIC GAGE CONTROL IN CONTINUOUS ROLLING

A functional diagram of a continuous cold-rolling mill is shown in Figure 1.1. The same mill, but without a coiler, is used in continuous hot rolling.

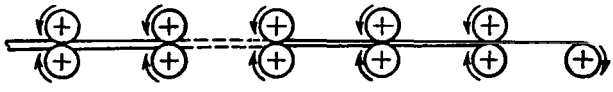

FIGURE 1.1. Schematic of a continuous rolling mill.

The roll mill stands are placed sequentially one after the other. Pressing screws on each stand alter the position of the top roll and thus adjust the clearance between the working rolls. The strip gage can be altered by changing the roll gap, as well as by raising the rolling tension (up to the yield point). Both these control techniques can be applied simultaneously. In automatic gage control, the pressing screws are regulated by a roll positioning system, whereas the tensile stress is adjusted by appropriately modifying the main drive velocities.

It is fairly obvious, however, that these two groups of control systems are interconnected through the rolled strip. A particularly pronounced interrelationship is observed in hot rolling mills, a fact which follows from various experimental data. The effect of stress on strip gage is evident from the bulging of the head and the tail of the piece, where the rolling tension is nil.

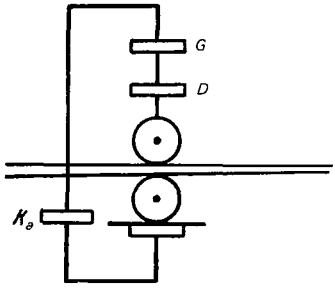

FIGURE 1.2. Block diagram of strip gage control in one of the mill stands.

Figure 1.2 is a block diagram of an automatic roll-gap control system for one of the mill stands. Similar systems are provided in each stand. Hydraulic dynamometers under the rolls act as thickness gages, and looper gears between the mill stands measure the stresses (not shown in Figure 1.2). For the sake of generality it is assumed that all the mill stands are equipped with hydraulic dynamometers and that stress measurements are taken between every two stands.

Let us consider the general control relationships for a rolling mill. The equation for the entire mill can be obtained by writing the equations for each stand with appropriate front and back tensions. *

Consider the equation of the i-th stand. The physical properties of continuous rolling are described by the following relations:

$$M_{si} = M'_{si} - \frac{D_i}{2j_i} f_{i,i+1} + \frac{D_i H_i}{2j_i H_{i-1}} f_{i,i-1}, \tag{1.1}$$

$$V_i = \gamma_i n_i [1 + b_i (f_{i,i+1} - f_{i,i-1})], \tag{1.2}$$

$$\Delta f_{i,i+1} = c_i \int_0^t (\Delta V'_{i,i+1} - \Delta V_i)\, dt, \tag{1.3}$$

where $\gamma = \frac{\pi D_i (1 + S_{0i})}{60 j_i}$, $c_i = \frac{EQ_i}{l_{i,i+1}}$.

In these equations

M_{si} = the reduced static torque in tension rolling,
M'_{si} = the static torque in tension-free rolling,
n_i = the velocity of the motor,
$f_{i,i+1}$ = interstand tension,
V'_i = strip velocity on entering the stand,
V_i = strip velocity on leaving the stand,
D_i = roll diameter,
j = motor-to-roll transmission ratio,
H_i = strip gage after the i-th stand,
Q_i = strip cross section,
E = modulus of elasticity of the rolled metal,
$l_{i,i+1}$ = interstand distance,
S_0 = forward creep in tension-free rolling,
b_i = forward-creep coefficient.

Under steady-state conditions the strip enters the i-th stand at the same velocity that it leaves the $(i-1)$-th stand.

From the constancy of the per-second volume in rolling we can find a dependence of the strip tension on the main drive velocities of the nearby stands. Assuming constant strip width, we find

$$V_{i-1} H_{i-1} = H_i V_i. \tag{1.4}$$

Inserting for V_i and V_{i-1} their expressions from (1.2) and solving (1.4) for n_i, we find

$$n_i = \frac{\gamma_{i-1} n_{i-1} H_{i-1} [1 + b_{i-1} (f_{i-1,i} - f_{i-2,i-1})]}{\gamma_i H_i [1 + b_i (f_{i,i+1} - f_{i-1,i})]}. \tag{1.5}$$

The differential equation of motion of the electric drive can be written in the form

$$\frac{GD^2}{375} \frac{dn_i}{dt} = M_{i\,m} - M_{i\,res} = M_{i\,m} - (M_{si} - M'_{si}). \tag{1.6}$$

The motor torque $M_{i\,m}$ is found from the relation

$$M_{i\,m} = C_{i\,i} \Phi_i I_{i\,a}. \tag{1.7}$$

* A rolling mill as a multivariable plant is considered by N. P. Druzhinin /13/ and A. A. Fel'dbaum /66/.

For a Ward – Leonard machine with constant exciting current we find

$$U_i = L_{i\,a}\frac{dI_{i\,a}}{dt} + R_{a\,i}I_{i\,a} + C_{ei}\Phi_i n_i, \tag{1.8}$$

whence

$$M_{i\,m} = C_{\tau\,i}\Phi_i \frac{U_i - C_{ei}\Phi_i n_i}{L_{i\,a}\frac{d}{dt} + R_{i\,a}}. \tag{1.9}$$

Substituting (1.9) in (1.6) we have

$$\frac{GD^2}{375}\frac{dn_i}{dt} = C_{\tau\,i}\Phi_i \frac{U_i - C_{ei}\Phi_i n_i}{L_{i\,a}\frac{d}{dt} + R_{i\,a}} - m_i, \tag{1.10}$$

where

$$m_i = M_{s\,i} - M'_{s\,i}.$$

The resistance torque depends on a number of factors. From the theory of plastic deformation /13/ the pressure on the rolls is given by

$$P = P[R_i, \Phi_i, \mu_i, F_{i1}, F_{i2}, H_{i1}, H_{i2}, K_i], \tag{1.11}$$

where P is a nonlinear function of the relevant parameters, R_i the effective roll radius, H_{i1} the ingoing gage for the i-th roll, H_{i2} the outgoing gage, F_{i1} the back tension on the strip, F_{i2} the front tension, Φ_i the contact arc, μ_i the friction coefficient.

The rolling torque is a function of the same variables and roll radius. It is expressed by another nonlinear function, thus:

$$M_i = M[R_i, R'_i, \Phi_i, \mu_i, F_{i1}, F_{i2}, H_{i1}, H_{i2}, K_i]. \tag{1.12}$$

For small increments of the variables in (1.11) and (1.12), assuming R', R, Φ, μ, and K to be constant, we may write

$$dP = \frac{\partial P_i}{\partial H_{i1}}dH_{i1} + \frac{\partial P_i}{\partial H_{i2}}dH_{i2} + \frac{\partial P}{\partial F_{i1}}dF_{i1} + \frac{\partial P}{\partial F_{i2}}dF_{i2} \tag{1.13}$$

and

$$dM_i = \frac{\partial M_i}{\partial H_{i1}}dH_{i1} + \frac{\partial M_i}{\partial H_{i2}}dH_{i2} + \frac{\partial M_i}{\partial F_{i1}}dF_{i1} + \frac{\partial M_i}{\partial F_{i1}}dF_{i2}. \tag{1.14}$$

Using lower-case letters for the small increments and constant coefficients for the partial derivatives, we write

$$\Delta P = K_{i1}h_{i1} + K_{i2}h_{i2} + K_{i3}f_{i1} + K_{i4}f_{i4}, \tag{1.15}$$
$$\Delta m = K_{i5}h_{i1} + K_{i6}h_{i2} + K_{i7}f_{i1} + K_{i8}f_{i2}, \tag{1.16}$$

where

$$K_{i1} = \frac{\partial P_i}{\partial H_{i1}}, \quad K_{i2} = \frac{\partial P_i}{\partial H_{i2}}, \quad K_{i3} = \frac{\partial P_i}{\partial F_{i1}}, \quad K_{i4} = \frac{\partial P_i}{\partial F_{i2}},$$
$$K_{i5} = \frac{\partial M_i}{\partial H_{i1}}, \quad K_{i6} = \frac{\partial M_i}{\partial H_{i2}}, \quad K_{i7} = \frac{\partial M_i}{\partial F_{i1}}, \quad K_{i8} = \frac{\partial M_i}{\partial F_{i2}}.$$

Equations similar to (1.15) and (1.16) can be derived for all the mill stands. In addition to the individual stand equations, there are also coupling equations which describe the continuous operation of the entire mill.

We use the following notation: primed quantities describe the state of the ingoing strip, lower-case letters denote small increments, and absolute values are represented by capital letters subscripted with a zero.

The increment of the loading torque in the i-th stand is written from (1.16) as

$$m_{id} = K_{i5}h'_i - K_{i6}h_i - K_{i7}f_i + K_{i8}f_{i-1}; \tag{1.17}$$

the continuity equation is

$$(V'_{i0} + v'_i)(H'_{i0} + h'_i) = (V_{i0} + v_i)(H_{i0} + h_i). \tag{1.18}$$

The change in strip tension due to elastic deformation is written as

$$\frac{df_i}{dt} = C_i(V'_{i+1} - v_i), \tag{1.19}$$

where C_i is a constant.

The velocity of the ingoing strip is higher than the linear velocity V_r of the roll surface. It is given by the relation

$$V = V_r(1 + S') = V_r(1 + S_0 + S) = V_r(1 + S_0)\left(1 + \frac{S}{1 + S_0}\right), \tag{1.20}$$

where S is the forward creep, dependent on strip tension, S_0 the forward creep in tension-free rolling. In the linear approximation the forward creep as a function of tension is given by the relation

$$S = b(1 + S_0)\Delta F. \tag{1.21}$$

From (1.20) and (1.21) we have

$$V = V_r(1 + S_0)(1 + b_0\Delta F) = \frac{\pi D_i N_m}{60j}(1 + S_0)[1 + b(F_i - F_{i-1})]. \tag{1.22}$$

Linearizing,

$$V_i = An_i + B(f_i - f_{i-1}), \tag{1.23}$$

where

$$A = \frac{\pi D_i(1 + S_0)}{60j} \quad \text{and} \quad B = AN_i.$$

A section of the rolled strip emerging from the given stand reaches the next stand after a certain time lag

$$\tau_i = \frac{l_i}{V_i}, \tag{1.24}$$

where l_i is the interstand distance, V_i the strip velocity. Thus,

$$h_i(t) = h_{i-1}(t - \tau_i), \tag{1.25}$$

and making use of (1.25) we write for (1.17)

$$m_{id\,i}=K_{i5}h_{i-1}(t-\tau_i)-K_{i6}h_i+K_{i8}f_{i-1}-K_{i7}f_i. \tag{1.26}$$

Let

$$\frac{GD^2}{375}\frac{R_a}{C_tC_e\Phi^2}=T_t,\quad \frac{L_a}{R_a}=T_a,\quad \frac{R}{C_tC_e\Phi^2}=K,\quad \frac{1}{C_e\Phi}=K_m.$$

We write the following two equations in Laplace transforms:

$$\begin{aligned}&[T_{i\,t}p(1+T_{i\,a}p)+1]\,n_i(p)=\\&=K_iU_{g\,i}(p)-K_i(1+T_{i\,a}p)\,K_{i5}e^{-\tau_ip}h_i(p)+K_i(1+T_{i\,a}p)\,K_{i6}h_i(p)-\\&\qquad-K_i(1+T_{i\,a}p)\,K_{i8}f_{i-1}(p)+K_i(1+T_{i\,a}p)\,K_{i7}f_i(p)\end{aligned} \tag{1.27}$$

and

$$pf_i(p)=C_iA_in_{i+1}(p)-C_iAn_i(p)+C_iB_if_{i+1}(p)+C_iB_if_{i-1}(p). \tag{1.28}$$

Substituting $f_i(p)$ from (1.28) in (1.27), we find

$$\begin{aligned}&[T_{t\,i}p^2(1+T_{a\,i}p)]\,n_i(p)=\\&\qquad=K_{i\,m}pU_{g\,i}-K_i(1+T_{i\,a}p)\,K_{i\,g}pe^{-\tau_ip}h_i(p)+\\&+K_ip(1+T_{i\,a}p)K_{i6}h_i(p)-K_i(p)(1+T_{i\,a}p)\,K_{i8}f_{i-1}(p)+\\&+K_i(1+T_{i\,a}p)\,K_{i7}C_iA_in_{i+1}(p)-K_i(1+T_{i\,a}p)\,K_{i\,7}C_iA_in_i(p)+\\&+K_i(1+T_{i\,a}p)\,C_iB_iK_{i7}f_{i+1}(p)+K_i(1+T_{i\,a}p)\,K_{i7}C_iB_if_{i-1}(p).\end{aligned} \tag{1.29}$$

After simple manipulations we have

$$\begin{aligned}&[T_{i\,t}p^2(1+T_{i\,a}p)+p+K_iK_{i7}C_iA_i(1+T_{i\,a}p)]\,n_i(p)=\\&=K_{i\,m}pU_{i\,m}(p)+K_i[K_{i6}p(1+T_{i\,a}p)-K_{i5}(1+T_{i\,a}p)\,pe^{-\tau_ip}]\,h_i(p)+\\&\qquad+K_iK_{i7}C_iA_i(1+T_{i\,a}p)\,n_{i+1}(p)+K_iC_iB_iK_{i7}f_{i+1}(p)+\\&\qquad\qquad+K_i[(1+T_{i\,a}p)(K_{i7}C_iB_i-pK_{i8})]\,f_{i-1}(p).\end{aligned} \tag{1.30}$$

As a final step in the derivation of the equation of main drive control, we have to choose an appropriate measuring device and to relate the main drive velocities to strip tension. Loopers are adequate measuring devices for continuous hot-rolling mills. In cold-rolling mills the tension can be found from motor load. Without going into this technical question, we assume that a suitable device is available for tension measurements. Then:

(a) tension between the $(i+1)$-th and i-th stands

$$f_{i+1}(p)=K_dW_d(p)\,[n_{ref}(p)-n_{i+1}(p)], \tag{1.31}$$

(b) tension between the $(i-1)$-th and i-th stands

$$f_{i-1}(p)=K_dW_d(p)\,[n_{ref}(p)-n_{i-1}(p)]. \tag{1.32}$$

The motor voltage $U_m(p)$ receives feedback from strip tension measurements. Let $W_d(p)$ be the transfer function of the measuring device and $K_aW_g(p)$ the transfer function of the generator, the exciter, and the amplifier

(if any); then

$$U_{m}(p) = W_{d}(p)\, W_{g}(p)\, K_{a}\,[n_{ref\,i}(p) - n_{i}(p)]. \tag{1.33}$$

Substituting (1.31), (1.32), and (1.33) in (1.30), we find

$$\begin{aligned}
&[T_{i\,i}p^{2}(1+T_{i_{a}}p)+p+K_{i}(1+T_{i_{z}}p)\,K_{i7}C_{i}A_{i}+\\
&\quad+K_{i\,m}pW_{d}(p)\,W_{g}(p)\,K_{a}]\,n_{i}(p)-[K_{i}K_{i7}C_{i}A_{i}(1+T_{i_{a}}p)-\\
&\quad-K_{i}C_{i}B_{i}K_{i7}K_{d\,i}(1+T_{i_{a}}p)\,W_{d\,i}(p)]\,n_{i+1}(p)+\\
&\quad+K_{i}[(1+T_{i_{a}}p)(K_{i7}C_{i}B_{i}-K_{i8}p)]\,K_{d}W_{d\,i}(p)\,n_{i-1}(p)+\\
&\quad+K_{i}[K_{i6}p(1+T_{i_{a}}p)-K_{i5}(1+T_{i_{a}}p)\,pe^{-\tau_{i}p}]\,h_{i}(p)=\\
&\qquad=W_{d\,i}(p)\,[K_{i\,m}pW_{g}(p)\,K_{a}+2K_{d}]\,n_{i\,ref}(p).
\end{aligned} \tag{1.34}$$

We see from equation (1.34) that the process of control in the i-th subsystem, where the controlled variable is n_i, is influenced by the controlled variables of subsystems $i-1$ and $i+1$. Each of these variables n_{i-1} and n_{i+1} has its own closed-loop control system. The various mill stands are described by a set of equations analogous to (1.34) with $i = 1, 2, \ldots, 6$.

One of the components of equation (1.34) — the term h_i — deserves special consideration. If the strip gage is controlled by tension alone, h_i is the external disturbance or load. In some instances of cold rolling h_i may therefore be considered as a load on the tension control system, whereby h_i is maintained between certain predetermined limits. In the general case, when the strip gage is controlled by simultaneously adjusting the tension and the reduction, special control subsystems are provided for h_i. The number of these subsystems is equal to the number of stands with reduction control. In cold rolling mills reduction is normally controlled in some three or four stands, and equations (1.34) are then supplemented by reduction control equations. If reduction control is instituted in only part of the stands, h_i remain in some of the equations in (1.34) as loads.

In continuous hot-rolling mills the gage is best regulated by appropriate reduction control; the tension should of course be maintained constant. Minimum tension is required, but it must be sufficient for strip centering. The process of gage control for a hot-rolled sheet can be investigated using equations (1.27), (1.28); these equations are solved for tension, which is presumably maintained constant.

Let us consider the equations that describe the controlled positioning of the pressing screws. The corresponding equations are equally applicable to both cold and hot rolling mills.

The screw positioning system has an actuator, a Ward – Leonard d. c. engine, say. Figure 1.2 is a schematic diagram of the roll-gap control system. The input is the reference gap value H_{ref}. The equation of the measuring device is

$$U_{out\ d\ i} = W_{dr\,i}(p)\,[H_{i\,ref} - H_{i}]. \tag{1.35}$$

The output of the measuring device is delivered to an amplifier and then to a generator. The equation of the amplifier and the generator is written as

$$U_{m\,i} = K_{a\,i}W_{g\,i}(p)\,U_{out\ d\ i}. \tag{1.36}$$

Now consider how the motor runs when the strip undergoes reduction. The torque equation is

$$\frac{GD_i^2}{375} pn = C_t \Phi_i I_i - M_{sr\,i}, \tag{1.37}$$

but the pressure on the rolls and the corresponding resistance torque sensed by the motor are given by

$$M_{sr\,i} = P_i r_{i\,eq}, \tag{1.38}$$

where P_i is defined by (1.11); $r_{i\,eq}$ is the equivalent arm which, together with the force P_i, produces the resistance torque on the motor.

By analogy with (1.14), we write

$$\Delta M_{sr\,i} = \frac{\partial M_{sr\,i}}{\partial H_{i1}} \Delta H_{i1} + \frac{\partial M_{sr\,i}}{\partial H_{i2}} \Delta H_{i2} + \frac{\partial M_{sr\,i}}{\partial F_{i1}} \Delta F_{i1} + \frac{\partial M_{sr\,i}}{\partial F_{i2}} \Delta F_{i2}, \tag{1.39}$$

and equation (1.37) takes the form

$$\frac{GD_{r\,i}^2}{375} pn = C_{tr\,i} \Phi_{r\,i} \frac{U_{m\,i} - C_{e\,r\,i} \Phi_{r\,i} n_{r\,i}}{L_{ar\,i} p + R_{ar\,i}} - \Delta M_{sr\,i}$$

or

$$\frac{GD_{r\,i}^2}{375} p^2 K_{r\,i} H_i = C_{t\,i} \Phi_i \frac{U_{m\,i} - C_{ei} \Phi_i n_{r\,i}}{L_{ar\,i} p + R_{ar\,i}} - K_{m\,i} \Delta H_{i1} - K_{i10} \Delta H_{i2} - K_{11} \Delta F_{i1} - K_{12} \Delta F_{i2}.$$

After simple manipulations, making use of (1.36) and (1.35), we find

$$[T_{tr\,i} p (1 + T_{ar\,i} p) p K_{r\,i} + K_{r\,i} p + K_{m\,i} K_{ar\,i} W_{gr\,i}(p) W_{dr\,i}(p)] H_i(p) = K_{m\,i} K_{ar\,i} W_{g\,i}(p) W_{dr\,i}(p) H_{ref}(p) - a_{r\,i} (1 + T_{ar\,i} p) [K_{m\,i} \Delta H_{1i} + K_{i10} \Delta H_2 + H_{11} \Delta F_{i1} + K_{i12} \Delta F_{i2}]. \tag{1.40}$$

The screw positioning equations for the other stands are obtained by assigning an appropriate value to the subscript i. For a three-stand system, $i = 1, 2, 3$.

In equation (1.40), H_i is in a sense H_{i-1} and $H_2 = H_{i+1}$. ΔF_1 and ΔF_2 are the coupling terms interconnecting equations (1.40) for $i = 1, 2, 3$ with equations (1.34) for $i = 1, 2, 3, 4, 5, 6$.

We have thus obtained two sets of equations: one describing the positioning of pressing screws and the other main drive control. Jointly these equations describe, in the linear approximation, the dynamics of gage control by simultaneous regulation of roll gap and rolling tension.

§ 1.2. A COMPLEX POWER SYSTEM AS A MULTI-VARIABLE CONTROLLED OBJECT

By a complex power system we mean a quite general configuration of power generating stations in a grid of arbitrary load. Each power station individually is a complex system comprising a few or even a few dozen powerful synchronous generators and other equipment. For the sake of simplicity each station is replaced in our analysis by an equivalent

synchronous generator and an equivalent prime mover. Numerous studies /46, 47, 69, 70, etc./ have shown that this substitution is fully permissible in many practically significant cases. It is further assumed that each equivalent generator is excitation-controlled and the equivalent prime mover (a steam or hydraulic turbine) is provided with speed control. Furthermore, all the machines except the first have secondary frequency control. Figure 1.3 is a block diagram of one element in a complex power system which comprises n equivalent units (prime mover and generator). We will derive an equation of the system for the case of small deviations of the controlled variables from a preset operating mode. The active and the reactive impedances in the system are assumed constant during each particular transient.

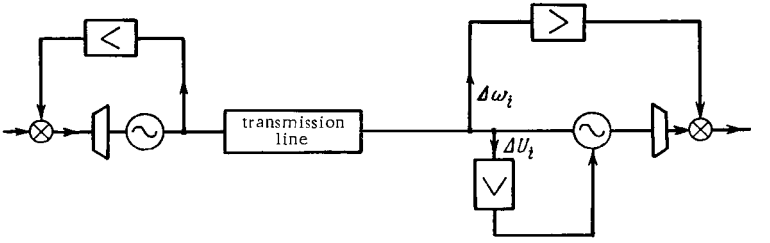

FIGURE 1.3. An i-th unit of a complex power system.

We start with the equations of the various components of the i-th equivalent unit.

1. The equation of motion of the i-th equivalent unit is

$$J_i \frac{d\,\Delta\omega_i}{dt} = \Delta M_i, \tag{1.41}$$

where J_i is the reduced moment of inertia of the unit, $\Delta\omega_i$ the change in frequency, ΔM_i the torque increment.

The torque is made up of two components: the actuating torque and the resistance (generator) torque; we may thus write

$$\Delta M_i = \Delta M_{ii} + \Delta M_{ia}, \tag{1.42}$$

where ΔM_{ii} is the change in generator torque, ΔM_{ia} the change in actuating torque.

The resistance torques are expressed by the functional dependence

$$M_i = M_i(\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}, E_{di1}, \ldots, E_{din}, \omega_1, \omega_2, \ldots, \omega_n) \tag{1.43}$$

and

$$\Delta M_{ii} = -\sum_{k=1}^{n} \frac{\partial M_i}{\partial \delta_{ik}} \Delta\delta_{ik} - \sum_{k=1}^{n} \frac{\partial M_i}{\partial E_{dik}} \Delta E_{ik} - \sum_{k=1}^{n} \frac{\partial M_i}{\partial \omega_k} \Delta\omega_k, \tag{1.44}$$

where δ_{ik} are the phase angles between the free-running e.m.f. of the k-th generator and the voltage developed by the i-th generator (which is

regarded as the leading generator), E_{di} is the free-running e.m.f. of the i-th generator, $\Delta\omega_k$ is the change in frequency for the k-th generator. If the first generator is regarded as the leading generator, the phase angles in the equation for any i-th generator should be reckoned from the e.m.f. vector of the first.

All partial derivatives in (1.44) are found from the corresponding static characteristics. Replacing them with appropriate constant coefficients and making use of (1.42) and (1.44), we write (1.41) in the form

$$J_i \frac{d\,\Delta\omega_i}{dt} = -\sum_{k=1}^{n} \alpha_{ik}\,\Delta\delta_{ik} - \sum_{k=1}^{n} \beta_{ik}\,\Delta E_k - \sum_{k=1}^{n} \gamma_{ik}\,\Delta\omega_k + \Delta M_{ia}$$

or

$$J_i \frac{d\,\Delta\omega_i}{dt} + \gamma_{ii}\,\Delta\omega_i = -\sum_{k=1}^{n} \alpha_{ik}\,\delta_{ik} - \sum_{k=1}^{n} \beta_{ik}\,\Delta E_k - \sum_{\substack{k=1 \\ k\neq i}}^{n} \gamma_{ik}\omega_k + \Delta M_{ia}. \qquad (1.45)$$

Here

$$\alpha_{ik} = \frac{\partial M_i}{\partial \delta_{ik}}, \quad \beta_{ik} = \frac{\partial M_i}{\partial E_{dik}}, \quad \gamma_{ik} = \frac{\partial M_i}{\partial \omega_k}.$$

2. The equations for the phase angles δ_{ik} are

$$\Delta\delta_{ik} = \int (\Delta\omega_i - \Delta\omega_k)\,dt$$

or

$$\frac{d\,\Delta\delta_{ik}}{dt} = \Delta\omega_i - \Delta\omega_k. \qquad (1.46)$$

Equation (1.46) clearly remains valid when the e.m.f. vectors of all the machines are reckoned from the e.m.f. vector of the first machine.

3. We now derive the equation of the motor's excitation circuit. It is assumed that the synchronous generator is excited by a special exciter. The fast electromagnetic processes in the stator circuit of the synchronous generator are ignored at this stage.

The transient in the rotor circuit of the i-th generator is described by the following differential equation:

$$E_{ide} = E_{id} + T_{id0} \frac{dE'_{id}}{dt}, \qquad (1.47)$$

where T_{id0} is the time constant of the excitation circuit, E'_{id} is the e.m.f. across the synchronous reactive impedance.

If no voltage control is provided, E_{ide} is constant. In voltage-controlled generators E_{ide} depends on the parameters of the excitation circuit, the exciter, and the voltage control system. For small deviations equation (1.47) takes the form

$$\Delta E_{ide} = \Delta E_{id} + T_{id0} \frac{d\,\Delta E'_{id}}{dt}. \qquad (1.48)$$

4. Let $W_{0i}(p)$ be the transfer function of the exciter and voltage controller. The relationship between ΔE_{ide} and the change in voltage at the generator

terminals is then written as

$$\Delta E_{ide}(p) = W_{0i}(p)\,\Delta U_g(p). \tag{1.49}$$

The voltage at the generator terminals is measured, and not the free-running e.m.f. Another expression is therefore required, relating the free-running e.m.f. to the voltage at the terminals.

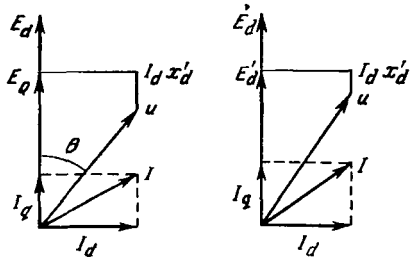

FIGURE 1.4. Vector diagrams of a synchronous machine.

From the vector diagram (Figure 1.4) we may write

$$\bar{E}'_{id} = \bar{E}_{id} - (x_{id} - x'_{id})\bar{I}_{id} \tag{1.50}$$

and

$$\bar{U}_i^2 = (\bar{E}_{id} - \bar{I}_{id}x_{id})^2 + \bar{I}_{iq}^2 x_{iq}^2. \tag{1.51}$$

Here x'_{id} is the transient reactive impedance of the generator, x_{id} the reactive impedance of the generator along the longitudinal axis, x_{iq} the reactive impedance along the transverse axis, I_{id} the longitudinal component of the stator current, I_{iq} the transverse component of the stator current.

We further assume for the sake of generality that the generator being considered is a salient-pole machine; then for the current components we write*

$$I_{id} = \frac{\bar{E}_{Qi}}{Z_{ii}}\cos\alpha_{ii} - \frac{\bar{E}_{id}}{Z_{ii}}\cos(-\delta_{i1} - \alpha_{ii}), \tag{1.52}$$

$$\bar{I}_{iq} = \frac{\bar{E}_{Qi}}{Z_{ii}}\sin\alpha_{ii} + \frac{E_{id}}{Z_{ii}}\sin(-\delta_{i1} - \alpha_{ii}). \tag{1.53}$$

Here $\bar{E}_{Qi}$ is the equivalent e.m.f. of salient-pole generators:

$$\bar{E}_{Qi} = A\bar{E}_{id} + BE_{id}\cos(-\delta_{i1} - \alpha_{ii}). \tag{1.54}$$

The constants A and B are expressed in terms of the generator parameters:

$$A = \frac{1 + \dfrac{x_{iq} - x'_{id}}{x_{id} - x_{iq}}}{\dfrac{x_{id} - x'_{id}}{x_{id} - x_{iq}} + \dfrac{\cos\alpha_{ii}}{Z_{ii}}(x_{id} - x'_d)},$$

* Detailed derivation of these equations is given, e.g., in /47, 69/.

$$B=\frac{\frac{x_{id}-x'_{id}}{Z_{ii}}}{\frac{x_{id}-x'_{id}}{x_{id}-x_{iq}}+\frac{\cos\alpha_{ii}}{Z_{ii}}(x_{id}-x'_{d})},$$

where is the self-impedance of the substitution circuit between the i-th and the 1st generator (Figure 1.5), Z_{1i} is the mutual impedance of the system between these generators.

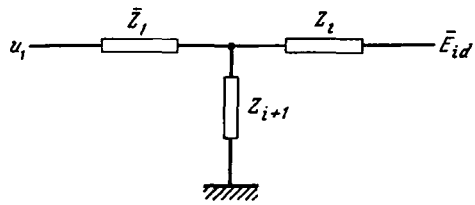

FIGURE 1.5. Diagram of impedances.

Substituting (1.52), (1.53), and (1.54) in (1.50) and (1.51) and linearizing, we find after simple manipulations

$$\Delta E'_{id}=V_i\,\Delta\delta_{i1}+Q_i\,\Delta E_{id}, \tag{1.55}$$

$$\Delta U_i=L_i\,\Delta\delta_{i1}+N_i\,\Delta E_{id}, \tag{1.56}$$

where

$$V_i=1-\frac{x_{id}-x'_{id}}{Z_{ii}}A\cos\alpha_{ii},$$

$$Q_i=(x_{id}-x'_{id})\,u_{10}\left(\frac{B\cos\alpha_{ii}}{Z_{ii}}-\frac{1}{Z_{i1}}\right)\sin(-\delta_{1i}-\alpha_{1i}),$$

$$L_i=\frac{U_{10}}{2U_{i0}}\left\{E_{id}\sin(-\delta_{1i}-\alpha_{1i})\left[2\,\frac{B\cos\alpha_{ii}}{Z_{ii}}-\frac{1}{Z_{i1}}\right]x_{id}-\right.$$

$$-\frac{2Ax_{id}^2\cos\alpha_{ii}}{Z_{ii}}\left(\frac{B\cos\alpha_{ii}}{Z_{ii}}-\frac{1}{Z_{i1}}\right)+$$

$$\left.+\sin 2(-\delta_{i1}-\alpha_{i1})\left[\frac{U_{10}x_{iq}^2}{Z_{i1}^2}-U_{10}x_{id}^2\left(\frac{B\cos\alpha_{ii}}{Z_{ii}}-\frac{1}{Z_{i1}}\right)\right]\right\},$$

$$N_i=\frac{1}{2U_{i0}}\left[2E_{id}\left(1-2x_{id}\,\frac{A}{Z_{i1}}\cos\alpha_{ii}+\frac{A^2}{Z_{ii}^2}\cos\alpha_{ii}x_{id}^2\right)-\right.$$

$$\left.-U_{10}\cos(-\delta_{i1}-\alpha_{i1})\left(\frac{B\cos\alpha_{ii}}{Z_{ii}}-\frac{1}{Z_{i1}}\right)\left(2x_{id}-2x_{id}^2\frac{\cos\alpha_{ii}}{Z_{ii}}\right)\right].$$

5. The equation of speed and frequency control for the i-th generator is

$$\Delta\mu_i=R_i(p)\,\Delta\omega_i. \tag{1.57}$$

We have thus obtained the following set of equations describing the i-th generalized unit:

$$(J_ip+\gamma_{ii})\,\Delta\omega_i=-\sum_{k=1}^{n}\alpha_{ik}\,\Delta\delta_{ik}-\sum_{k=1}^{n}\beta_{ik}\,\Delta E_k-\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}\,\Delta\omega_k+\Delta M_{i_a}, \tag{1.58a}$$

$$
\left.
\begin{aligned}
p\,\Delta\delta_{ik} &= \Delta\omega_i - \Delta\omega_k,\\
\Delta E_{ide} &= \Delta E_{id} + T_{id0}\,p\,\Delta E'_d,\\
\Delta E_{ide} &= W_{i0}(p)\,\Delta U_i,\\
\Delta E'_{id} &= V_i\,\Delta\delta_{ik} + Q_i\,\Delta E_{id},\\
\Delta U_i &= L_i\,\Delta\delta_{ik} + N_i\,\Delta E_{id},\\
\Delta M_i &= R_i(p)\,\Delta\omega_i.
\end{aligned}
\right\} \qquad (1.58b)
$$

These equations cannot be simplified unless we have decided what the controlled variables of the system are. As far as the quality of the generated electric power is concerned, the frequency and the voltage must be maintained constant. In some cases, however, stability considerations, say, suggest that the phase angle δ_{ik} should be controlled. This approach is also advisable if the voltage of the generalized units (power stations) is controlled via the phase angles δ_{ik} using remote phase meters. The controlled variables are therefore the frequency ω_i, the phase angle $\Delta\delta_{ik}$, the generator voltage U_i, the e.m.f. of all other generators E_k, and the frequency of the other generators $\omega_k (k \neq i)$.

Eliminating E'_{id}, E_{id}, and E_{ide} from (1.58), we obtain the following set of equations for the i-th unit:

$$
\left.
\begin{aligned}
(J_i p + \gamma_{ii})\,\omega_i &= -\sum_{k=1}^{n} \alpha_{ik}\,\Delta\delta_{ik} - \sum_{k=1}^{n} \beta_{ik}\,\Delta E_k - \sum_{\substack{k=1\\k\neq i}}^{n} \gamma_{ik}\,\Delta\omega_k + \Delta M_{ia},\\
p\,\Delta\delta_{ik} &= \Delta\omega_i - \Delta\omega_k,\\
T_{id0}\left(V_i - \frac{Q_i L_i}{N_i}\right) p\,\Delta\delta_{ik} &- \frac{L_i}{N_i}\,\Delta\delta_{ik} +\\
&+\left[T_{id0}\,\frac{Q_i}{N_i}\,p + \frac{1}{N_i} - W_{0i}(p)\right] U_i = 0,\\
\Delta M_i &= R_i(p)\,\Delta\omega_i,\\
\Delta M_{ia} &= K\,\Delta M_i.
\end{aligned}
\right\} \qquad (1.59)
$$

Similar sets of equations can be obtained for the other generalized units of the complex power system. The entire power system is described by equations (1.58) with $i = 1, 2, \ldots, n$. If some units have no frequency or voltage control, the corresponding terms vanish.

The coupling in this case is twofold. First, in each individual unit the generator voltage (or e.m.f.) is sensitive to variations of frequency and speed. Each unit thus constitutes a multivariable interacting system. But the coupling goes further: the processes in the i-th unit affect all the other units of the power system as a whole.

§ 1.3. A RECTIFYING COLUMN

A rectifying column is a very common installation in petrochemical and gas industries. From our point of view a rectifying column is a typical multivariable plant representative of a whole class of industrial processes adapted to automatic control. We therefore proceed with a discussion of some elementary properties essential for the understanding of the physical foundation of the rectification process and then give detailed mathematical treatment of some simple cases. Although there is a

considerable variety of rectifying columns, they all operate on the same principle and can be described by identical mathematical equations.

Rectification is a kind of distillation, i.e., separation of a liquid mixture into constituents which have different boiling points. Rectification is carried out in such a way that an ascending stream of vapor comes in contact with a descending countercurrent of condensed liquid, i.e., the base of the column is heated while its upper portion is cooled. A schematic diagram of a rectifying column is shown in Figure 1.6.

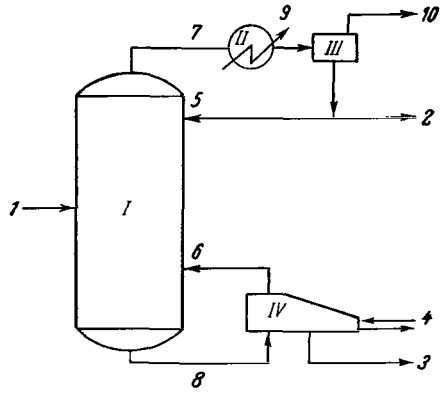

FIGURE 1.6. A rectifying column for the separation of a binary mixture:

I column, II condenser, III accumulator, IV reboiler; 1) crude feed, 2) overhead product, 3) bottoms, 4) vapor, 5) reflux, 6) vapor-liquid mixture, 7) vapor phase, 8) liquid phase, 9) water, 10) gas out.

The main element of a rectifying column is the packing, namely plates or trays on which the vapor comes in contact with the liquid phase. The vapor is thus enriched with the low-boiling component, and the proportion of the high-boiling component in the liquid also increases. A functional diagram of a bubble-cap plate is shown in Figure 1.7.

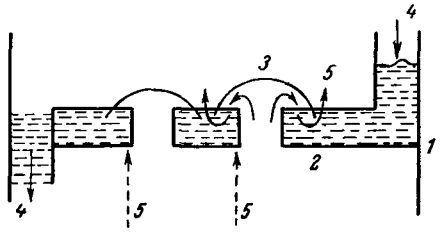

FIGURE 1.7. Functional diagram of the rectification process:

1) column wall, 2) plate, 3) cap, 4) liquid, 5) vapor.

Depending on the composition of the crude feed, we distinguish between columns for separation of binary mixtures and columns for separation of multicomponent mixtures. The calculations for multicomponent rectifying columns are substantially more complicated, and the corresponding processes have been poorly studied.

(a) COLUMNS FOR SEPARATION OF MULTICOMPONENT MIXTURES

A binary column is that where the finished product is only the overhead distillate or the bottoms. Automation of binary rectifying columns should be implemented with due regard to the industrial objectives and the engineering aspects of the process. The following cases can be distinguished.

Case 1. Product concentration higher than required. Losses less than permissible.

The goal is to make the product as pure as possible and to produce as much of it as is feasible, irrespective of power requirements.

Case 2. Product concentration higher than required. Optimum power consumption.

A very-high-purity product is to be separated, but its quantity is determined by power losses from cooling water and vapor.

Case 3. Product concentration not lower than the stipulated figure. Uniform product efflux.

The distillate constitutes a feed to another industrial process, so that excessive fluctuations of product discharge are undesirable.

Case 4. Optimum economy irrespective of product concentration and quantity.

The input and output variables of the rectifying unit illustrated in Figure 1.6 are shown to first approximation in Figure 1.8. Note that some of the input variables in Figure 1.8 are interrelated. For example, a change in the quantity of feed affects the condenser operation and the reflux temperature is altered; a change in the pumping rate of the overhead product alters the quantity of reflux, etc.

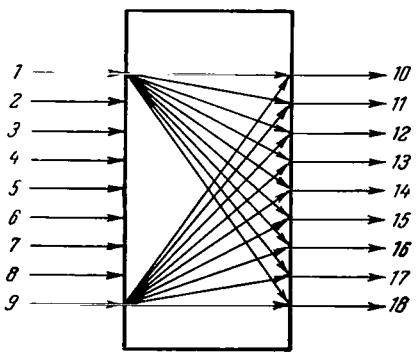

FIGURE 1.8. Schematic diagram of the variables in a binary column:

1) quantity of feed, 2) composition of feed, 3) temperature of feed, 4) reflux flow rate, 5) reflux temperature, 6) pumping of overhead product, 7) pumping of bottoms, 8) water flow rate in the condenser, 9) vapor flow rate in the reboiler, 10) top plate temperature, 11) bottom plate temperature, 12) k-th plate temperature, 13) composition of overhead product, 14) composition of bottoms, 15) composition of mixture on the bottom plate, 16) liquid level in the accumulator, 17) liquid level in the reboiler, 18) pressure in the column.

A simple rectifying unit for the separation of binary mixtures is thus a multivariable plant with numerous inputs and outputs. Complete description of the column requires knowledge of the relationships between the inputs and the outputs shown in Figure 1.8.

One of the main paths is the "feed composition-to-product concentration". Analytical and experimental (using laboratory rectifying units) studies of this path were published by various authors /83, 84/.

The equation of each plate is derived proceeding from the material balance of the low-boiling component. Under certain assumptions it has the form

$$(T_k p + 1) C_k = K_{1k} C_{k-1} + K_{2k} C_{k+1}, \tag{1.60}$$

where k is the plate number, K_{1k}, K_{2k} the gains, T_k a time constant, C_k concentration deviation of the low-boiling component on the k-th plate.

The equations of the condenser, reboiler, and feed plate differ only in the number of terms entering the right-hand side of (1.60). The constants T_k, K_{1k}, and K_{2k} depend on the velocity of vapor and liquid streams, the form of the equilibrium curve interrelating the composition of the vapor and the liquid phase on each plate, and the liquid mass on the plate.

Equations (1.60) ignore the hydrodynamics of vapor and liquid streams. This omission is rectified with the aid of the equations

$$\left.\begin{aligned}(\tau_1 p + 1) L_k &= L_{k+1},\\ (\tau_2 p + 1) V_k &= V_{k-1},\end{aligned}\right\} \tag{1.61}$$

where V_k is the flow rate of vapor rising from the k-th plate, L_k the flow rate of liquid dripping from the k-th plate, τ_1 and τ_2 are the corresponding time constants of the k-th plate.

Putting $k = 1, \ldots, n$, we obtain a set of equations for this simple binary column. We wish to stress again that the equations were obtained proceeding from the material balance of one of the components.

(b) COLUMNS FOR SEPARATION OF BINARY MIXTURES

Rectifying towers for fractionation of petroleum products are much more difficult to control than the previously considered simple binary columns. As we have noted, distillation in binary columns is mostly described by three inequalities:

$$\left.\begin{aligned}C_{bt} &\leqslant C_{bt\,3},\\ C_{tb} &\leqslant C_{tb\,3},\\ D &\geqslant D_3,\end{aligned}\right\} \tag{1.62}$$

where C_{bt} is the content of the bottoms component in the overhead distillate, C_{tb} is the content of the overhead product in the bottoms, D the separation factor. The subscript 3 denotes the standard reference values of the corresponding quantities.

A vacuum distillation column for multicomponent mixtures is described by considerably more numerous constraints. The more obvious of these are the following:

$$\left.\begin{aligned} T_{lb} &\geqslant T_{lb\,3},\\ T_{eb} &\leqslant T_{eb\,3},\\ T_{flash} &> T_{flash\,3},\\ V &< V_3,\\ C &> C_3,\\ D &\geqslant D_3, \end{aligned}\right\} \qquad (1.63)$$

where T_{lb} is the lower boiling temperature of the fraction, T_{eb} is the end boiling temperature, T_{flash} the flash-point temperature, V the viscosity of the fraction, C the color of the fraction, D the separation factor.

This set of constraints is of course applicable to each withdrawn fraction.

A characteristic feature of a vacuum distillation column from the point of view of a control engineer is that its optimum operating conditions are characterized by a generalized index which is a functional of numerous controlled variables (reference values and other quantities). Optimal reference values are determined by the industrial plant conditions. If the distillate is a marketable product, optimization is impossible without knowing the dependence of cost and market price on product composition.

An optimality test is provided, say, by the profit amassed in time T. If the dependence of profit on product composition is a function with an extremum, the column is optimized if a maximum profit is ensured. If the distillate requires further processing before it can be marketed, we must know the relationship between distillate composition and the cost of subsequent processing. It is thus clear that constraints (1.63) constitute only the first step in the development of optimum control systems for rectifying towers. However, in general, as the constraints (1.63) approach equalities, the operation of the column under the given set of conditions becomes progressively more economic.

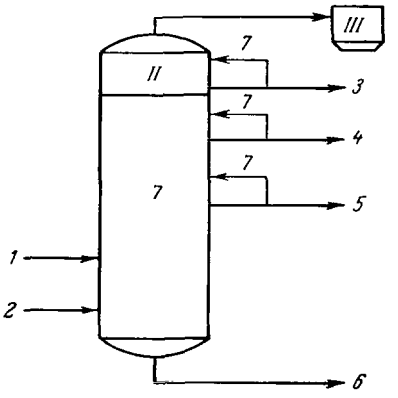

FIGURE 1.9. Column for separation of a multicomponent mixture:

I column, II accumulator, III barometric condenser; 1) feed, 2) superheated vapor, 3) 1st fraction, 4) 2nd fraction, 5) 3rd fraction, 6) bottoms, 7) reflux.

The static and the dynamic characteristics of the column are required for the solution of the problem before us. In what follows we derive an equation relating the mass flow of the feed and the product to temperature conditions in the column. This statement of the problem is understandable since in most rectifying towers temperature is one of the controlled variables.

A technological diagram of the column is shown in Figure 1.9. The tapped products are the 2nd and 3rd fractions and the residuum.

The automatic control of these columns constitutes a complicated problem. With binary columns no more than two or three constraints had to be satisfied (e.g., concentration greater than reference, losses less than reference), while in vacuum distillation columns the number of constraints is much greater.

The five principal constraints for each fraction are the following:

1. Lower boiling point higher than reference.
2. End boiling point less than reference.
3. Viscosity less than reference.
4. Flash-point temperature higher than reference.
5. Color stronger than reference.

The "controllability" of the column thus becomes a very topical question.

The interrelationships between the column inputs and outputs are indicated in Figure 1.10, which shows only the most important variables. Block diagrams of the rectifying tower are given in Figures 1.11 and 1.12.

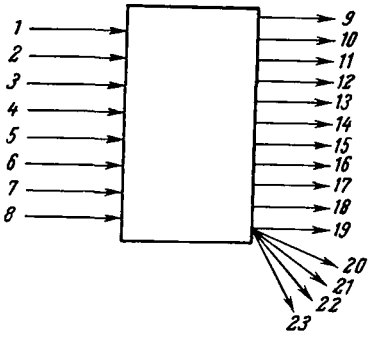

FIGURE 1.10. Illustrating the inputs and outputs of a multicomponent column:

1) feed flow rate, 2) feed temperature, 3) feed viscosity, 4) reflux of 2nd fraction, 5) withdrawal of 2nd fraction, 6) reflux of 3rd fraction, 7) withdrawal of 3rd fraction, 8) vapor flow rate, 9) lower boiling point of 2nd fraction, 10) end boiling point of 2nd fraction, 11) viscosity of 2nd fraction, 12) flash-point temperature of 2nd fraction, 13) color of 2nd fraction, 14, 15, 16, 17, 18) ditto for 3rd fraction, 19) temperature of 2nd fraction, 20) temperature of 3rd fraction, 21) temperature on k-th plate, 22) liquid level in the accumulator, 23) bottoms quality.

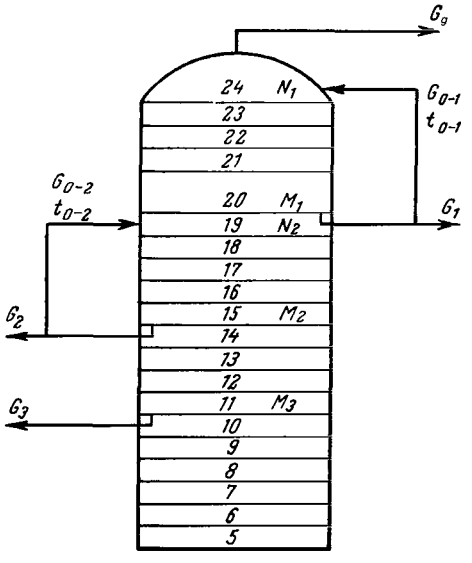

FIGURE 1.11. Illustrating the derivation of equations for a multivariable column.

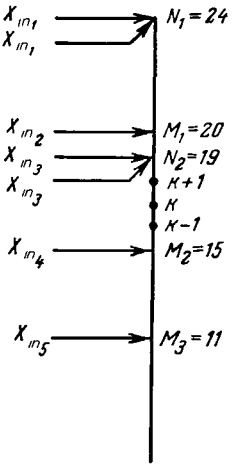

FIGURE 1.12. Illustrating the derivation of equations for a multivariable column.

Nomenclature:

G_1, G_2, G_3 = the withdrawn quantities of each fraction;
$G_{0\text{-}1}$, $G_{0\text{-}2}$ = the quantities of reflux for the corresponding fractions;
G_g = gas flow into the barometric condenser;
k = plate number, reckoned from bottom to top;
M_l = number of plate from which the fraction is withdrawn;
N_l = number of the reflux plate;
$X_{\text{in}\,l}$ = deviation in mass flow of feed;
$X^*_{\text{in}\,l}$ = deviation in temperature of feed;
X_k = deviation in temperature of liquid phase on the k-th plate;
h_k = liquid level deviation on the k-th plate;
t_k = temperature of liquid phase on k-th plate;
T_k = temperature of vapor phase on k-th plate;
$t_{0\text{-}l}$ = temperature of reflux;
L_k = flow of liquid dripping from k-th plate;
v_k = flow of vapor rising from k-th plate;
c_{liq} = slope factor of the straight line approximating the temperature dependence of the specific heat of liquid;
c_v = slope factor of the straight line approximating the temperature dependence of the specific heat of vapor;
m = mass of liquid on the plate;
H = level of liquid on the plate;
F = accumulator surface area;
D = column diameter;
ρ = liquid density;
δ = symbol of deviation.

Assumptions adopted in the derivation of equations

1. The feed is liquid at its boiling point.
2. Temperature variation on the plates does not affect the velocity of the vapor.
3. Vapor and liquid temperature deviations on the k-th plate are related by

$$\delta T_k = k\,\delta t_k.$$

4. Total condensation occurs on plate N_1 ($G_g = 0$).
5. The delay of the vapor on the k-th plate is negligible.
6. Change in level is negligible on all plates, except M_l.
7. $L_{M_1} = 0$, since in this column the downpour from the 20th plate is quenched.
8. The effect of water steam flow on column temperature is negligible.
9. The hydrodynamics of the liquid is ignored.

In the mathematical part we use the well-known equations of heat balance. We consider several cases.

Case 1. The equation of the k-th plate.

The equation of statics (steady-state conditions, see Figure 1.13) can be written in the following form:

$$v_{k-1}c_v T_{k-1} - v_k c_v T_k + L_{k+1}c_{lq}t_{k+1} - L_k c_{lq}t_k = 0.$$

The equation of dynamics:

$$\begin{aligned} v_{k-1}c_v(T_{k-1}+\delta T_{k-1}) - v_k c_v(T_k+\delta T_k) + \\ +(L_{k+1}+\delta L_{k+1})c_{lq}(t_{k+1}+\delta t_{k+1}) - \\ -(L_k+\delta L_k)c_{lq}(t_k+\delta t_k) = A\frac{dt_k}{d\tau}, \end{aligned}$$

where $A = mc_{lq}$.

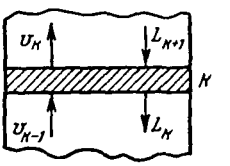

FIGURE 1.13. Illustrating the derivation of equations for a multivariable column.

Seeing that the liquid flow in the given section may change only due to a change in the quantity of reflux, we may write

$$\delta L_k = \delta L_{k+1} = \delta L_{k-1} \sim X_{in\,3}.$$

In view of assumption 3 above

$$\delta T_k = k\,\delta t_k \sim kX_k.$$

Passing to an equation in deviations, linearizing, and Laplace-transforming, we find

$$(a_k p + 1)X_k = b_k X_{k-1} + c_k X_{k+1} + k_{3k}X_{in\,3}, \qquad (1.64)$$

where

$$a_k = \frac{A}{c_{lq}h_k + c_v k v_k}$$ the time constant of the k-th plate,

$$\left.\begin{aligned} b_k &= \frac{c_v k v_{k-1}}{c_{lq}L_k + c_v k v_k}, \\ c_k &= \frac{c_{lq}L_{k+1}}{c_{lq}L_k + c_v k v_k} \end{aligned}\right\}$$ nondimensional gain factors,

$$k_{3k} = \frac{c_{lq}(t_{k+1}-t_k)}{c_{lq}L_k + c_v k v_k}$$ dimensional gain factor.

Equation (1.64) is sometimes conveniently rewritten as

$$X_k + \beta_k X_{k-1} + \sigma_k X_{k+1} = \gamma_{3k}X_{in\,3}, \qquad (1.65)$$

where

$$\beta_k = -\frac{b_k}{a_k p + 1}, \quad \sigma_k = -\frac{c_k}{a_k p + 1}, \quad \gamma_{3k} = \frac{k_{3k}}{a_k p + 1}.$$

Case 2. The equation of the N_2-th plate (Figure 1.14):

$$v_{N_2-1} c_v (T_{N_2-1} + \delta T_{N_2-1}) - v_{N_2} c_v (T_{N_2} + \delta T_{N_2}) -$$
$$- (L_{N_2} + \delta L_{N_2}) c_{lq} (t_{N_2} + \delta t_{N_2}) + (G_{0\text{-}2} + \delta G_{0\text{-}2}) c_{lq} (t_{0\text{-}2} + \delta t_{0\text{-}2}) = A \frac{dt_{N_2}}{dt}.$$

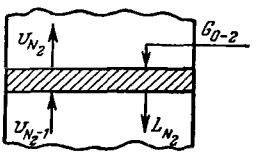

FIGURE 1.14. Illustrating the derivation of equations for a multivariable column.

Acting as before and seeing that

$$\delta t_{0\text{-}2} \sim X^*_{\text{in } 3}$$

and

$$\delta G_{0\text{-}2} = \delta L_{N_2} \sim X_{\text{in } 3},$$

we find

$$(a_{N_2} p + 1) X_{N_2} = b_{N_2} X_{N_2-1} + k_{3N_2} X_{\text{in } 3} + k^*_{3N_2} X^*_{\text{in } 3}, \tag{1.66}$$

where

$$a_{N_2} = \frac{A}{c_{lq} L_{N_2} + c_v k v_{N_2}},$$
$$b_{N_2} = \frac{c_v k v_{N_2-1}}{c_{lq} L_{N_2} + c_v k v_{N_2}},$$
$$c_{N_2} = 0,$$
$$k_{3N_2} = \frac{c_{lq} (t_{0\text{-}2} - t_{N_2})}{c_{lq} L_{N_2} + c_v k v_{N_2}},$$
$$k^*_{3N_2} = \frac{c_{lq} G_{0\text{-}2}}{c_{lq} L_{N_2} + c_v k v_{N_2}}.$$

Case 3. The equation of the M_2-th plate (Figure 1.15). The equation of dynamics:

$$v_{M_2-1} c_v (T_{M_2-1} + \delta T_{M_2-1}) - v_{M_2} c_v (T_{M_2} + \delta T_{M_2}) +$$
$$+ (L_{M_2+1} + \delta L_{M_2+1}) c_{lq} (t_{M_2+1} + \delta t_{M_2+1}) - (L_{M_2} + \delta L_{M_2}) c_{lq} (t_{M_2} + \delta t_{M_2}) -$$
$$- (G_2 + \delta G_2 + G_{0\text{-}2} + \delta G_{0\text{-}2}) c_{lq} (t_{M_2} + \delta t_{M_2}) = A \frac{dt_{M_2}}{dt}.$$

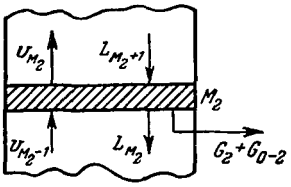

FIGURE 1.15. Illustrating the derivation of equations for a multivariable column.

Acting as before and seeing that

$$\delta G_{0\text{-}2}=\delta L_{M_3+1}\sim X_{\text{in }3}$$

and

$$\delta G_2=-\delta L_{M_3}\sim X_{\text{in }4},$$

we find

$$(a_{M_2}p+1)X_{M_2}=b_{M_2}X_{M_2-1}+c_{M_2}X_{M_2+1}+k_{3M_2}X_{\text{in }3}, \tag{1.67}$$

where

$$a_{M_2}=\frac{A}{c_{lq}(G_2+G_{0\text{-}2}+L_{M_2})+c_v k v_{M_2}},$$

$$b_{M_2}=\frac{c_v k v_{M_2-1}}{c_{lq}(G_2+G_{0\text{-}2}+L_{M_2})+c_v k v_{M_2}},$$

$$c_{M_2}=\frac{c_{lq}L_{M_2+1}}{c_{lq}(G_2+G_{0\text{-}2}+L_{M_2})+c_v k v_{M_2}},$$

$$k_{3M_2}=\frac{c_{lq}(t_{M_2+1}-t_{M_2})}{c_{lq}(G_2+G_{0\text{-}2}+L_{M_2})+c_v k v_{M_2}}.$$

Case 4. The equation of the k-th plate ($M_2 > k > M_3$). The equation is derived precisely as in Case 1. Seeing that

$$\delta L_k\sim -X_{\text{in }4},$$

we obtain

$$(a_k p+1)X_k+b_k X_{k-1}+c_k X_{k+1}+k_{4k}X_{\text{in }4}, \tag{1.68}$$

where

$$k_{4k}=\frac{c_{lq}(t_k-t_{k+1})}{c_{lq}L_k+c_v k v_k}.$$

Case 5. The equation of the M_1-th plate (Figure 1.16). The equation of dynamics:

$$\begin{aligned}v_{M_1-1}c_v(T_{M_1-1}+\delta T_{M_1-1})-v_{M_1}c_v(T_{M_1}+\delta T_{M_1})+\\+(L_{M_1+1}+\delta L_{M_1+1})c_{lq}(t_{M_1+1}+\delta t_{M_1+1})-\\-(G_{0\text{-}1}+\delta G_{0\text{-}1}+G_1+\delta G_1)c_{lq}(t_{M_1}+\delta t_{M_1})=\\=c_{lq}F\rho\frac{d[(H+\delta H)(t_{M_1}+\delta t_{M_1})]}{dt}.\end{aligned}$$

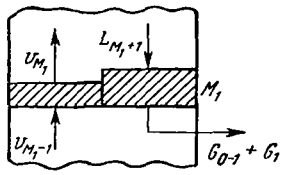

FIGURE 1.16. Illustrating the derivation of equations for a multivariable column.

Passing to the equation in deviations, linearizing, and Laplace-transforming, we make use of the fact that

$$\delta L_{M_i+1} = \delta G_{0-1} \sim X_{in\,1}$$

and

$$\delta G_1 \sim X_{in\,2}$$

to obtain

$$c_{lq} t_{M_i} F \rho p h + [Ap + (G_{0-1} + G_1) c_{lq} + v_{M_i} c_v k] X_{M_i} = \\ = v_{M_i-1} c_v k X_{M_i-1} + L_{M_i+1} c_{lq} X_{M_i+1} + c_{lq} (t_{M_i+1} - t_{M_i}) X_{in\,1} - t_{M_i} c_{lq} X_{in\,2}. \quad (1.69)$$

The equation of material balance for this plate is

$$F \rho p h = - X_{in\,2}, \quad (1.70)$$

i.e., equation (1.68) in fact is a combination of two independent equations, (1.69) and (1.70):

$$[Ap + (G_{0-1} + G_1) c_{lq} + v_{M_i} c_v k] X_{M_i} = \\ = v_{M_i-1} c_v k X_{M_i-1} + L_{M_i+1} c_{lq} X_{M_i+1} + c_{lq} (t_{M_i+1} - t_{M_i}) X_{in\,1}. \quad (1.71)$$

From (1.69) and (1.70) it follows that the liquid level in the accumulator is independent of temperature, and the withdrawal of the distillate does not affect the temperature. These conclusions follow from the linearized equations; the validity of the starting linearized equation, however, requires experimental verification.

The equations for the other plates are derived similarly to one of the cases (1.64) – (1.69).

An analysis of the equations of various plates has shown that the general equation of the k-th plate may be written as

$$(a_k p + 1) X_k = b_k X_{k-1} + c_k X_{k+1} + k_{1k} X_{in\,1} + k^*_{1k} X^*_{in\,1} + \\ + k_{3k} X_{in\,3} + k^*_{3k} X^*_{in\,3} + k_{4k} X_{in\,4} + k_{5k} X_{in\,5}. \quad (1.72)$$

Some of the coefficients in (1.72) may be zero.

The coefficients in (1.72) are the following:

$a_k = \frac{A}{c_{lq} Q_k + c_v k v_k}$, the plate time constant, Q_k being the total flow of liquid from the plate;

$$b_k = \frac{c_v k v_{k-1}}{c_{lq} Q_k + c_v k v_k}, \quad c_k = \frac{c_{lq} L_{k+1}}{c_{lq} Q_k + c_v k v_k}$$

} nondimensional gain factors taking into account the temperature of the overlying and underlying plates;

k_{1k}, k_{3k} are coefficients corresponding to the effect of the reflux mass flow; the coefficients k_{ik} ($i = 2, 3$) are defined by the following relations:

(a) for the plate receiving the reflux

$$k_{ik} = \frac{c_{lq}(t_0 - t_k)}{c_{lq} Q_k + c_v k v_k};$$

(b) for any plate in the reflux section, including the plate from which the distillate is withdrawn,

$$k_{ik} = \frac{c_{lq}(t_{k+1} - t_k)}{c_{lq} Q_k + c_v k v_k};$$

(c) for other plates $k_{ik} = 0$;

k^*_{1k} and k^*_{3k} are coefficients that allow for the effect of reflux temperature; the coefficients k^*_{ik} ($i = 1, 3$) are defined by the following relations:

(a) for the sprinkled plate

$$k^*_{ik} = \frac{c_{lq} G_0}{c_{lq} Q_k + c_v k v_k};$$

(b) for other plates $k^*_{ik} = 0$;

k_{4k} and k_{5k} are coefficients that allow for distillate outflow; the coefficients k_{ik} ($i = 4, 5$) are defined by the following relations:

(a) for the plate from which the distillate is withdrawn $k_{ik} = 0$;

(b) for any underlying plate $k_{ik} = \frac{c_{lq}(t_k - t_{k+1})}{c_{lq} Q_k + c_v k v_k}$.

Dividing the left- and the right-hand sides of (1.72) by $(a_k p + 1)$, we find

$$\beta_k X_{k-1} + X_k + \sigma_k X_{k+1} = \gamma_{1k} X_{in\,1} + \gamma^*_{1k} X^*_{in\,1} + \gamma_{3k} X_{in\,3} + \gamma^*_{3k} X^*_{in\,3} + \gamma_{4k} X_{in\,4} + \gamma_{5k} X_{in\,5}. \quad (1.73)$$

The equation relating the various column variables is then written as

$$B \cdot \bar{X} = D \bar{X}_{in}, \quad (1.74)$$

where B and D are matrices,

$$B = \begin{Vmatrix} \dots & \dots & \dots & \dots & \beta_{24} & 1 \\ \dots & \dots & \dots & \beta_{23} & 1 & \sigma_{23} \\ \dots & \dots & \beta_{22} & 1 & \sigma_{22} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \beta_3 & 1 & \sigma_3 & \dots & \dots \\ \beta_2 & 1 & \sigma_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \sigma_1 & \dots & \dots & \dots & \dots \end{Vmatrix},$$

$$D=\begin{Vmatrix} \gamma_{1,24} & \gamma^{*}_{1,24} & 0 & 0 & 0 & 0 \\ \gamma_{1,23} & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{1,20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{3,19} & \gamma^{*}_{3,19} & 0 & 0 \\ 0 & 0 & \gamma_{3,18} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \gamma_{3,15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{4,14} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \gamma_{4,10} & \gamma_{5,10} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix};$$

$\bar{X}$ and $\bar{X}_{in}$ are column vectors,

$$\bar{X}=\begin{Bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_{23} \\ X_{24} \end{Bmatrix} \quad \text{and} \quad \bar{X}_{in}=\begin{Bmatrix} X_{in\,1} \\ \cdot \\ \cdot \\ X_{in\,5} \end{Bmatrix}.$$

§ 1.4. OIL STRATA WITH LINEAR SEEPAGE

From the point of view of multivariable control, oil fields have much in common with the controlled objects discussed in the previous sections. Anticipating, we can say that the common feature for these multivariable objects is that quality is regarded as a generalized index dependent on a variety of factors and numerous constraints, so that the control problem is reduced to extremizing some functional. It will be shown in Chapter Eight that, under certain additional conditions, the control of an oil field can be reduced to extremization of a linear form.

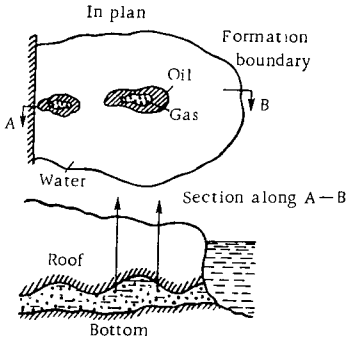

FIGURE 1.17. Schematic diagram of an oil reservoir.

Crude oil is a mixture of solid, liquid, and gaseous hydrocarbons impregnating a porous medium. If a well is sunk in this medium, the stratal pressure will drive the crude oil to the surface.

In order to maintain sufficient stratal pressure in the production well, water is pumped into the reservoir through so-called injection wells which ensure what is known as secondary recovery of oil. Figure 1.17 is a schematic diagram of an oil reservoir. The output, or controlled variable for each i-th well is the quantity of liquid Q_i produced. Note that the well may produce stratal water as well as oil, and the yield therefore does not provide an unambiguous quality criterion of well operation. The problem of efficient working of an oil field will be considered in Chapter Eight. Here we will only derive the control equation of the reservoir, taking Q_i as the well output.

The oil field may have two operating modes:

(a) elastic, when the pressure at any point in the pay rock is a function of time, other conditions being constant. This is a transient mode, arising immediately after a certain disturbance is applied to the pay, e.g., when the well is stopped;

(b) rigid, when the pressure at any point is constant during a certain time interval, being dependent on the position of that point only.

In the general case of elastic or rigid conditions, one of the main problems of the theory is to determine the pressure at any point in the oil-bearing stratum and at the face of the well at any time; the size and the physico-geological characteristics of the field are assumed to be known.

It is shown in the literature /72, 75, and others/ that the general behavior of an oil reservoir is described by the following partial differential equation:

$$\frac{\partial}{\partial x}\left(\frac{kh}{\mu}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{kh}{\mu}\frac{\partial P}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{kh}{\mu}\frac{\partial P}{\partial z}\right)=F(x,\ y,\ z)+\frac{1}{a^2}\frac{\partial P}{\partial t}\ldots, \qquad (1.75)$$

where P is the stratal pressure, h the thickness of the stratum, k the permeability, μ the viscosity of the medium, $a^2=\frac{k}{\mu\beta^*}$, where β^* is the storage coefficient of the stratum, or the so-called piezopermeability, $\frac{\mu}{kh}=R_{hyd}$ is the hydraulic resistance of the medium, $F(x, y, z)$ a discontinuity function, which is identically zero at all points of the reservoir, with the exception of the points at which wells are sunk.

For the rigid mode $\frac{\partial P}{\partial t}=0$ and (1.75) takes the form

$$\frac{\partial}{\partial x}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial z}\right)=F(x,\ y,\ z). \qquad (1.76)$$

The problem can be simplified if planar conditions are assumed, i.e., the thickness of the oil stratum is regarded as small in comparison with its extent. The flow of liquid along the z axis can be ignored so that $\frac{\partial P}{\partial z}=0$. Equations (1.75) and (1.76) thus take the form

$$\frac{\partial}{\partial x}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial y}\right)=F(x,\ y)+\frac{1}{a^2}\frac{\partial P}{\partial t}, \qquad (1.77)$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{R_{hyd}}\frac{\partial P}{\partial y}\right)=F(x,\ y). \qquad (1.78)$$

With boundary conditions of the first kind the pressure

$$P_b=f(x,\ y,\ z)$$

on the boundary is constant, and the pressure drop is thus zero,

$$\Delta P_b=0; \qquad (1.79)$$

alternatively, the rate of change of the pressure drop on the boundary is zero (boundary conditions of second kind), thus:

$$\frac{\partial (\Delta P_b)}{\partial t} = 0.$$

The last case corresponds to a closed oil reservoir.

Solving equation (1.75) or (1.77) for appropriate boundary and initial conditions, we obtain the debit Q as a function of pressure. The problem thus reduces to finding $P = F_1(x, y, z, t)$ and $Q = F_2(x, y, z, t)$. For various producing conditions the debit of the well can be expressed by the following relation, which is in fact the Darcy law of filtration:

$$Q = \frac{\Delta P}{R(x, y, z)}, \tag{1.80}$$

where $R(x, y, z)$ is the equivalent resistance to liquid flow in a pressure gradient ΔP.

If for the first well $\frac{1}{R(x, y, z)} = a_{11}$, and there are no other wells, equation (1.80) takes the form

$$Q_1 = a_{11} \Delta P_1. \tag{1.81}$$

Consider the case of an oil reservoir with n production wells. A change in operating conditions in any of the wells causes redistribution of pressure in the entire field. For the rigid mode, the behavior of the field is described by the equations

$$\left.\begin{aligned} a_{11}Q_1 + a_{12}Q_2 + \ \cdots\ + a_{1n}Q_n &= \Delta P_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}Q_1 + a_{n2}Q_2 + \ \cdots\ + a_{nn}Q_n &= \Delta P_n, \end{aligned}\right\} \tag{1.82}$$

where a_{ij} is a coefficient that describes to what extent the processes in the i-th well influence those in the j-th well. Equations (1.82) describe the behavior of an oil reservoir in the rigid mode from the standpoint of multivariable control theory.

In the elastic mode the stratal processes are described by convolution integrals. In what follows, however, we are only concerned with optimization of rigid operating conditions, and equations (1.82) are thus quite sufficient.

Chapter Two

MULTIVARIABLE CONTROL SYSTEMS WITH BASIC ELEMENTS

§ 2.1. INTRODUCTORY REMARKS

In this chapter we consider automatic control systems where each single-variable loop is built of basic elements only. By basic elements we mean /39/ the controlled object or plant, the measuring device or transducer, the controller or regulator and, in general, a number of amplifiers. Simple systems of this kind are designed for each controlled variable, and schematically they are represented by single-loop diagrams.

As we have previously noted, the relationship (or coupling) between controlled variables in multivariable systems may be attributed to the peculiar properties of the controlled plant. In this case we say that the controlled variables are interrelated through the controlled object (or through its properties). An alternative way of saying it is that the variables are plant-coupled. The relationship between the controlled variables may be also artificially introduced by means of transducers or control paths; finally, some interrelation may be imposed by the technological or production process. In what follows, the term multivariable control systems (MCS) is understood in the quite general sense of systems with interconnected variables, irrespective of the particular mode of coupling. From the examples considered in Chapter One we see that the number of controllers or regulators is not always equal to the number of controlled variables. If the controlled variables are regarded as the plant outputs and the controller coordinates as the inputs, we may assume quite generally that the number of outputs is less than or equal to the number of inputs. Study of simple multivariable control systems with single-loop subsystems should provide a foundation for the design of effective control systems, a problem of obvious practical importance. In order to simplify the mathematical description of the process, we shall first consider the properties of multivariable plants.

A multivariable plant may take on two fairly general alternative configurations shown in Figures 2.1 and 2.2. For the sake of simplicity, the transfer functions for two controlled variables only are shown. In the sequel the particular results for the two-variable system will be generalized without difficulty to any number of controlled variables. We do not consider here the case when the output of the coupling element $W_{ik}(p)$ is delivered neither to the input nor to the output of the element with the transfer function $W_{ii}(p)$, but to some intermediate point, since it is easily reduced to one of the principal cases by a simple modification of the function $W_{ik}(p)$.

We now proceed to derive an equation for the first controlled variable $Y_{1\,out}$ in cases depicted in Figures 2.1 and 2.2. For Figure 2.1 we have

$$Y_{1\,out}(p) = W_{11}(p)[X_{1\,in}(p) - W_{12}(p)Y_{2\,out}(p)] = \\ = W_{11}(p)X_{1\,in}(p) - W_{11}(p)W_{12}(p)Y_{2\,out}(p). \tag{2.1}$$

A similar equation can be written for the second channel. Let us now consider the second case, that in Figure 2.2:

$$Y_{1\,out}(p) = W_{11}(p)X_{1\,in}(p) + W_{12}(p)X_{2\,in}. \tag{2.2}$$

If the number of controlled variables is not two but n, the equation for the i-th controlled variable in the first configuration is

$$Y_{i\,out}(p) = W_{ii}(p)X_i(p) - W_{ii}(p)\sum_{\substack{k=1 \\ k\neq i}}^{n} W_{ik}(p)Y_{k\,out}(p), \tag{2.3}$$

and the output of the second configuration is

$$Y_{i\,out}(p) = W_{ii}(p)X_i(p) + \sum_{\substack{k=1 \\ k\neq i}}^{n} W_{ik}(p)X_{k\,in}(p). \tag{2.4}$$

The difference between the two alternatives is the following: in the first configuration the i-th output is dependent on the i-th input and the outputs of all the other controlled variables, whereas in the second

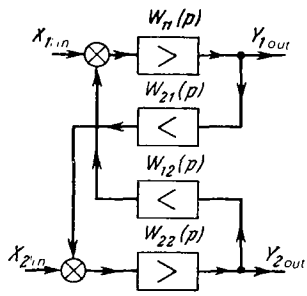

FIGURE 2.1. A plant with cross coupling.

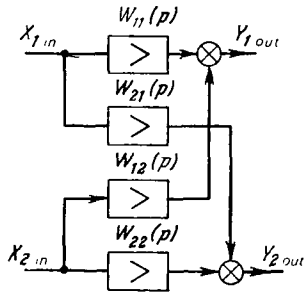

FIGURE 2.2. A plant with direct coupling.

configuration the i-th output is a function of the i-th input and all the other inputs. It is easily understood that the first case can be reduced to the second by a certain modification of the transfer function $W_{ik}(p)$. As we have not imposed any restrictions on the form of the coupling transfer function, we will consider the first configuration only (a system with cross coupling), using the general symbol $\alpha_{ik}(p)$ for the coupling coefficients. In the case of cross coupling, we obviously have

$$\alpha_{ik}(p) = W_{ik}(p); \tag{2.5}$$

and for direct coupling (Figure 2.2)

$$\alpha_{ik}(p) = \frac{W_{ik}(p)}{W_{kk}(p)}. \tag{2.6}$$

The controlled variables are often interconnected simultaneously by both direct coupling and cross coupling. This, however, does not alter the structure of equation (2.4). It is only the function $\alpha_{ik}(p)$ that changes. This approach to plant equations is justified because in practice the controlled object is fixed from the start and we are not free to change its structure. As regards the control system, the aim of the designer is to choose the optimum structure, and one does not generally start with equations of known form. In the sequel we therefore concentrate on methods of selection of control-system structures.

§ 2.2. TRANSFER FUNCTIONS OF MULTIVARIABLE CONTROL SYSTEMS WITH BASIC ELEMENTS

Consider a multivariable control system with n controlled variables interrelated through the controlled object. A subsystem made of basic-element components is provided for each of the controlled variables.

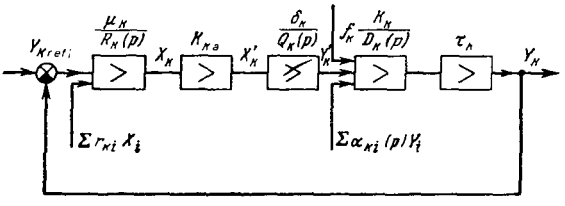

FIGURE 2.3. A general block diagram of a multivariable control system with basic elements.

(a) We assume that the measuring elements (transducers) are also interrelated (the case of load coupling will be considered under (b)). Figure 2.3 is a block diagram of the subsystem for the k-th controlled variable. The nomenclature pertaining to the k-th controlled variable:

K_k = the plant gain;
$D_k(p)$ = the denominator of the plant transfer function, henceforth called the self-operator;*
Y_k = the controlled variable;
τ_k = the loop delay (lag);
$Y_{k\,\mathrm{ref}}$ = the reference value of the controlled variable;
$\alpha_{ik}(p)$ = the coupling coefficient of the i-th and k-th variables, dependent on the properties of the plant: $\alpha_{ik}(p)$ is either a constant (positive or negative) or a function of the operator p;

* [In this translation the adjectival prefix "self-" qualifies quantities and expressions pertaining to an isolated single-variable subsystem which does not interact with the subsystems of other, "extraneous" variables.]

μ_k = the transducer gain;
$R_k(p)$ = the transducer self-operator;
Y'_k = the controller output;
$K_{k\,a}$ = the amplifier gain;
δ_k = the controller gain;
$Q_k(p)$ = the controller self-operator;
r_{kl} = a coupling coefficient between k-th and i-th transducers;
f_k = the load.

We now write the set of equations in Laplace transforms for the k-th controlled variable with zero initial conditions. * Making use of the nomenclature in Figure 2.3, we write the plant equation

$$D_k(p)\,e^{\tau_k p}Y_k(p)=K_k\left[-\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_{ki}(p)\,Y_i(p)+Y'_p(p)+f_k(p)\right], \tag{2.7}$$

the equation of the measuring device

$$R_k(p)\,X_k(p)=\mu_k\left[Y_{\mathrm{ref}\,k}(p)-Y_k(p)+\sum_{\substack{i=1\\ i\neq k}}^{n}r_{ki}X_i(p)\right], \tag{2.8}$$

the amplifier equation

$$X'_k(p)=K_{k\,a}X_k(p); \tag{2.9}$$

and the controller equation

$$Q_k(p)Y'_k(p)=\delta_k X'_k(p). \tag{2.10}$$

Eliminating $Y'_k(p)$, $X_k(p)$ and $X'_k(p)$ between (2.7), (2.8), (2.9), and (2.10), we obtain

$$\begin{aligned}&[D_k(p)\,R_k(p)\,Q_k(p)\,e^{\tau_k p}+K_kK_{k\,a}\delta_k\mu_k]\,Y_k(p)+\\&+K_kR_k(p)\,Q_k(p)\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_{ki}(p)\,Y_i(p)=K_kK_{k\,a}\delta_k\mu_kY_{k\,\mathrm{ref}}(p)+\\&+K_kK_{k\,a}\delta_k\mu_k\sum_{\substack{i=1\\ i\neq k}}^{n}r_{ki}X_i(p)+K_kR_k(p)\,Q_k(p)\,f_k(p).\end{aligned} \tag{2.11}$$

The subscript k runs from 1 to n, and we obtain a complete set of equations describing the behavior of the multivariable control system under the given conditions.

Putting $r_{ki}=0$ in (2.11) (the measuring devices are uncoupled), we obtain an equation for the class of MCS in which the controlled variables are interrelated through the controlled object only:

$$\begin{aligned}&[D_k(p)\,R_k(p)\,Q_k(p)\,e^{\tau_k p}+K_kK_{k\,a}\delta_k\mu_k]\,Y_k(p)+\\&+K_kR_k(p)\,Q_k(p)\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_{ki}(p)\,Y_i(p)=K_kK_{k\,a}\delta_k\mu_kY_{k\,\mathrm{ref}}(p)+\\&+K_kR_k(p)\,Q_k(p)\,f_k(p)\qquad(k=1,\ \ldots,\ n).\end{aligned} \tag{2.12}$$

* This means that initially the outputs of the various subsystems are zero, provided that they are described by first-order equations; if they are described by second-order equations, the first derivatives are also zero, etc. As regards the delay element, the output and its derivatives are assumed zero in the interval $(-\tau, 0)$.

As a particular example, the equation of a multidimensional servosystem can be derived from (2.11). A multidimensional servosystem is a MCS in which the controlled variables are interconnected through the measuring device only. Therefore putting in (2.11) $\alpha_{ki} = 0$, we find

$$[D_k(p) R_k(p) Q_k(p) e^{\tau_k p} + K_k K_{k\,a} \delta_k \mu_k] Y_k(p) =$$
$$= K_k K_{k\,a} \delta_k \mu_k Y_{k\,\mathrm{ref}}(p) + K_k K_{k\,a} \delta_k \mu_k \sum_{\substack{i=1 \\ i \neq k}}^{n} r_{ki} X_i(p) +$$
$$+ K_k R_k(p) Q_k(p) f_k(p) \qquad (k = 1, \ldots, n). \tag{2.13}$$

where

$$X_i = Y_{\mathrm{ref}\,i} - Y_i.$$

(b) In the preceding we have considered the interdependence of the controlled variables contributed by the properties of the controlled object and by the coupling between the measuring devices (an artificially introduced factor). In this case the load in the k-th control loop affects the controlled variables in all the other loops via the k-th controlled variable. In some cases, however, a change in the load in the k-th subsystem may directly influence some other controlled variables. It is moreover significant that the load (or the disturbance) is often introduced as an additional control factor. In these so-called combined control systems the proportional deviation control (the Watt-Polzunov principle) is combined with load control (Poncelet principle). The equation of a combined control system is obtained if equation (2.11) is modified to allow for load coupling. In a particular case, a combined control system may degenerate into a MCS with load coupling, provided that the load coupling is not employed as a control factor.

Let $\beta_{ki}(p)$ be a coefficient describing the effect of the i-th load on the k-th controlled variable; $\beta_{ki}(p)$ is a constant number or a function of the operator p. We assume that disturbances from extraneous loads (i.e., those not associated directly with the k-th variable) are also fed to the plant input. In this general case, we have

$$[D_k(p) Q_k(p) R_k(p) e^{\tau_k p} + K_{k\,\mathrm{tot}}] Y_k(p) +$$
$$+ K_k R_k(p) Q_k(p) \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_{ki}(p) Y_i(p) + K_{k\,\mathrm{tot}} \sum_{\substack{i=1 \\ i \neq k}}^{n} r_{ki}(p) Y_i(p) =$$
$$= K_{k\,\mathrm{tot}} Y_{k\,\mathrm{ref}}(p) + K_{k\,\mathrm{tot}} \sum_{\substack{i=1 \\ i \neq k}}^{n} r_{ki}(p) Y_{\mathrm{ref}\,i}(p) + K_k R_k(p) Q_k(p) P_k(p) +$$
$$+ K_k R_k(p) Q_k(p) \sum_{\substack{i=1 \\ i \neq k}}^{n} \beta_{ki}(p) f_i(p), \tag{2.14}$$

where

$$K_{k\,\mathrm{tot}} = K_k K_{k\,a} \mu_k \delta_k \qquad (k = 1, 2, \ldots, n).$$

If the disturbance from the self-loads is not delivered to the plant input but to the input of some other element in the control system, the function of p before the sum $\sum \beta_{ki}(p)$ will change, while the equation as a whole will retain its structure.

The set of equations (2.14) applies to the most general case of multivariable control systems, provided that the individual variables are controlled by single-loop systems. The equation of an ordinary combined control system (with a single controlled variable) can be obtained from (2.14) by putting $\alpha_{ih}(p) = r_{ih} = 0$.

§ 2.3. EQUATIONS OF MULTIVARIABLE CONTROL SYSTEMS IN MATRIX FORM

The equations describing MCS dynamics can be conveniently written in matrix form, which is very compact and sometimes facilitates the mathematical analysis of the system.

We use the following symbols: $a_{ii}(p)$ denotes the operatorial expressions preceding the self-variables in equation (2.14), $a_{hi}(p)$ denotes the operatorial expressions representing the influence of the i-th variable on the k-th variable. Then

$$a_{kk}(p) = D_k(p) R_k(p) Q_k(p) e^{\tau_k p} + K_k K_{k\,a} \delta_k \mu_k,$$
$$a_{ki}(p) = K_k R_k(p) Q_k(p) \alpha_{ki}(p) + K_k \mu_k \delta_k K_{a\,k} r_{ki},$$
$$g_{kk}(p) = K_k R_k(p) Q_k(p).$$

We also put

$$K_k K_{k\,a} \delta_k \mu_k = K_{k\,\mathrm{tot}},$$
$$K_k R_k(p) Q_k(p) \beta_{ki}(p) = b_{ki}(p), \quad K_{k\,\mathrm{tot}} r_{ki} = c_{ki}.$$

In this notation, equations (2.14) take the form

$$AY = (K_{\mathrm{tot}} + C) Y_{\mathrm{ref}} + DF + BF, \tag{2.15}$$

where

$$\left.\begin{aligned} A &= \begin{Vmatrix} a_{11}(p) & a_{12}(p) & \dots & a_{1n}(p) \\ a_{21}(p) & a_{22}(p) & \dots & a_{2n}(p) \\ \dots & \dots & \dots & \dots \\ a_{i1}(p) & a_{i2}(p) & \dots & a_{in}(p) \\ \dots & \dots & \dots & \dots \\ a_{n1}(p) & a_{n2}(p) & \dots & a_{nn}(p) \end{Vmatrix}, \quad Y = \begin{Vmatrix} Y_1(p) \\ Y_2(p) \\ \vdots \\ Y_i(p) \\ \vdots \\ Y_n(p) \end{Vmatrix}, \\ DF &= \begin{Vmatrix} g_{11}(p) f_1(p) \\ g_{22}(p) f_2(p) \\ \dots \\ g_{ii}(p) f_i(p) \\ \dots \\ g_{nn}(p) f_n(p) \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & b_{12}(p) & \dots & b_{1n}(p) \\ b_{21}(p) & 0 & \dots & b_{2n}(p) \\ \dots & \dots & \dots & \dots \\ b_{i1}(p) & b_{i2}(p) & \dots & b_{in}(p) \\ \dots & \dots & \dots & \dots \\ b_{n1}(p) & b_{n2}(p) & \dots & 0 \end{Vmatrix}, \\ F &= \begin{Vmatrix} f_1(p) \\ f_2(p) \\ \vdots \\ f_n(p) \end{Vmatrix}, \quad K_{\mathrm{tot}} Y_{\mathrm{ref}} = \begin{Vmatrix} K_{1\,\mathrm{tot}} Y_{1\,\mathrm{ref}} \\ K_{2\,\mathrm{tot}} Y_{2\,\mathrm{ref}} \\ \dots \\ K_{n\,\mathrm{tot}} Y_{n\,\mathrm{ref}} \end{Vmatrix}, \quad C = \begin{Vmatrix} 0 & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & 0 & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & c_{n3} & \dots & 0 \end{Vmatrix}. \end{aligned}\right\} \tag{2.15a}$$

Some particular cases of the general equation (2.15) are given in the following:

(a) the equation of an ordinary multivariable control system (coupling through the controlled object only)

$$A_0 Y_0 = K_{tot} Y_{ref} + DF; \tag{2.16}$$

(b) the equation of a multidimensional servosystem (coupling through the measuring elements only, i.e., $\alpha_{ki}(p) = \beta_{ki}(p) = 0$)

$$A_m Y = K_{tot} Y_{ref} + CY_{ref} + DF, \tag{2.17}$$

where

$$A_m = \begin{Vmatrix} a_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & a_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & a_{nn} \end{Vmatrix}; \tag{2.18}$$

(c) the equation of a control system with load coupling ($\alpha_{ki}(p) = r_{ki}(p) = 0$)

$$A_k Y = K_{tot} Y_{ref} + DF + BF, \tag{2.19}$$

where

$$A_k = \begin{Vmatrix} a_{11} & 0 & 0 \dots 0 \\ 0 & a_{22} & 0 \dots 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \dots a_{nn} \end{Vmatrix}$$

is a diagonal matrix.

An interesting particular case is that of controlled variables with identical control subsystems and symmetric coupling, i.e., $a_{ii}(p) = a_{kk}(p)$ and $\alpha_{ik}(p) = \alpha_{ki}(p)$. The matrix A is symmetric in this case, and the matrix A_k may be written as

$$A_k = a(p) E, \tag{2.20}$$

where

$$a(p) = a_{11}(p) = a_{22}(p) \dots = a_{nn}(p)$$

and

$$E = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 \end{Vmatrix} \tag{2.21}$$

is the identity matrix.

From (2.15), (2.16), (2.17), and (2.19) we obtain the respective matrix equations for the different cases.

The general case:

$$Y = A^{-1}[K_{tot} Y_{ref} + DF + BF + CY_{ref}]. \tag{2.22}$$

The case of an ordinary multivariable system:

$$Y = A_0^{-1}[K_{tot}Y_{ref} + DF]. \tag{2.23}$$

Multidimensional servosystem:

$$Y = A_m^{-1}[K_{tot}Y_{ref} + CY_{ref} + DF]. \tag{2.24}$$

Load-coupled control system:

$$Y = A_k^{-1}[K_{tot}Y_{ref} + DF + BF]. \tag{2.25}$$

In order to obtain equations in Laplace transforms, the inverse matrices A^{-1}, A_m^{-1}, A_k^{-1} should be found in explicit form for each controlled variable. We know from matrix theory that the inverse of a matrix is found in the following way:

1. The given matrix is transposed, i.e., its rows and columns are interchanged.
2. Each element of the transpose is replaced with its minor.
3. Each element in the matrix from 2 is divided by the determinant value of the system.
4. Each element of the matrix from 3 is assigned the sign $(-1)^{i+j}$, where i is the row number and j the column number of that element.

We now proceed to determine the inverse A^{-1}. First we write the transpose

$$A_t = \begin{Vmatrix} a_{11}(p) & a_{21}(p) & a_{31}(p) \dots a_{n1}(p) \\ a_{12}(p) & a_{22}(p) & a_{32}(p) \dots a_{n2}(p) \\ a_{13}(p) & a_{23}(p) & a_{33}(p) \dots a_{n3}(p) \\ \dots & \dots & \dots \\ a_{1l}(p) & a_{2l}(n) & \dots \quad \dots a_{nl}(p) \\ \dots & \dots & \dots \\ a_{1n}(p) & a_{2n}(p) & \dots \quad \dots a_{nn}(p) \end{Vmatrix} \tag{2.26}$$

and the determinant

$$\Delta = \begin{vmatrix} a_{11}(p) & a_{12}(p) \dots a_{1n}(p) \\ a_{21}(p) & a_{22}(p) \dots a_{2n}(p) \\ \dots & \dots \\ a_{l1}(p) & a_{l2}(p) \dots a_{ln}(p) \\ \dots & \dots \\ a_{n1}(p) & a_{n2}(p) \dots a_{nn}(p) \end{vmatrix}. \tag{2.27}$$

The minors of the elements of the transpose (2.26) with appropriate algebraic signs (the so-called cofactors) are denoted by $A_{ij}(p)$; here $A_{ij}(p)$ is the determinant of the transpose with the i-th row and the j-th column crossed out, and its sign is $(-1)^{i+j}$.

The inverse A^{-1} is thus written in the form

$$A^{-1} = \frac{1}{\Delta}\begin{Vmatrix} A_{11}(p) & -A_{21}(p) & \dots (-1)^{n+1}A_{n1}(p) \\ -A_{12}(p) & A_{22}(p) & \dots (-1)^{n+2}A_{n2}(p) \\ \dots & \dots & \dots \\ (-1)^{i+1}A_{1l}(p) & (-1)^{l+2}A_{2l}(p) & \dots (-1)^{l+n}A_{nl}(p) \\ \dots & \dots & \dots \\ (-1)^{n+1}A_{n1}(p) & (-1)^{n+2}A_{n2}(p) & \dots \quad A_{nn}(p) \end{Vmatrix}. \tag{2.28}$$

The matrices A_m^{-1} and A_k^{-1} are obtained similarly.

We now write in explicit form the expressions in brackets in (2.22). In our nomenclature,

$$K_{\text{tot}}Y_{\text{ref}} + DF = \begin{Vmatrix} K_{1\,\text{tot}}Y_{1\,\text{ref}}(p) + g_{11}(p)\,f_1(p) \\ K_{2\,\text{tot}}Y_{2\,\text{ref}}(p) + g_{22}(p)\,f_2(p) \\ \cdots\cdots\cdots\cdots \\ K_{n\,\text{tot}}Y_{n\,\text{ref}}(p) + g_{nn}(p)\,f_n(p) \end{Vmatrix}, \tag{2.29}$$

$$BF = \begin{Vmatrix} b_{11}(p)\,f_1(p) + b_{12}(p)\,f_2(p) + \cdots + b_{1n}(p)\,f_n(p) \\ b_{21}(p)\,f_1(p) + b_{22}(p)\,f_2(p) + \cdots + b_{2n}(p)\,f_n(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n1}(p)\,f_1(p) + b_{n2}(p)\,f_2(p) + \cdots + b_{nn}(p)\,f_n(p) \end{Vmatrix}, \tag{2.30}$$

$$CY_{\text{ref}} = \begin{Vmatrix} 0 + c_{12}(p)\,Y_{2\,\text{ref}}(p) + c_{13}(p)\,Y_{3\,\text{ref}}(p) + \cdots + c_{1n}(p)\,Y_{n\,\text{ref}}(p) \\ c_{21}(p)\,Y_{1\,\text{ref}}(p) + 0 + c_{23}(p)\,Y_{3\,\text{ref}}(p) + \cdots + c_{2n}(p)\,Y_{n\,\text{ref}}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}(p)\,Y_{1\,\text{ref}}(p) + c_{n2}(p)\,Y_{2\,\text{ref}}(p) + \cdots + c_{n,n-1}(p)\,Y_{n-1\,\text{ref}}(p) + 0 \end{Vmatrix}. \tag{2.31}$$

Substituting from (2.28), (2.29), (2.30), and (2.31) in (2.22), we find

$$\begin{aligned} Y = \frac{1}{\Delta} & \begin{Vmatrix} A_{11}(p) & -A_{21}(p) \cdots (-1)^{1+n} A_{n1}(p) \\ -A_{12}(p) & A_{22}(p) \cdots (-1)^{2+n} A_{n2}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (-1)^{n+1} A_{1n}(p), & A_{2n}(p) \cdots A_{nn}(p) \end{Vmatrix} \times \\ & \times \left\{ \begin{Vmatrix} K_{1\,\text{tot}}Y_{1\,\text{ref}}(p) + g_{11}(p)\,f_1(p) \\ K_{2\,\text{tot}}Y_{2\,\text{ref}}(p) + g_{22}(p)\,f_2(p) \\ \cdots\cdots\cdots\cdots \\ K_{n\,\text{tot}}Y_{n\,\text{ref}}(p) + g_{nn}(p)\,f_n(p) \end{Vmatrix} + \right. \\ & + \begin{Vmatrix} b_{11}(p)\,f_1(p) + b_{12}(p)\,f_2(p) + \cdots + b_{1n}(p)\,f_n(p) \\ b_{21}(p)\,f_1(p) + b_{22}(p)\,f_2(p) + \cdots + b_{2n}(p)\,f_n(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{n1}(p)\,f_1(p) + b_{n2}(p)\,f_2(p) + \cdots + b_{nn}(p)\,f_n(p) \end{Vmatrix} + \\ & \left. + \begin{Vmatrix} 0 + c_{12}(p)\,Y_{2\,\text{ref}}(p) + \cdots + c_{1n}(p)\,Y_{n\,\text{ref}}(p) \\ c_{21}(p)\,Y_{1\,\text{ref}}(p) + 0 + \cdots + c_{2n}(p)\,Y_{n\,\text{ref}}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}(p)\,Y_{1\,\text{ref}}(p) + \cdots + c_{n,n-1}(p)\,Y_{n\,\text{ref}}(p) + 0 \end{Vmatrix} \right\}. \end{aligned} \tag{2.32}$$

Multiplying, we obtain for the matrix of the controlled variables

$$\begin{aligned} Y = \frac{1}{\Delta} & \left\{ \begin{Vmatrix} \sum_{i=1}^{n} (-1)^{i+1} A_{i1}(p) \left[K_{i\,\text{tot}}Y_{i\,\text{ref}}(p) + g_{ii}(p)\,f_i(p)\right] \\ \sum_{i=1}^{n} (-1)^{i+2} A_{i2}(p) \left[K_{i\,\text{tot}}Y_{i\,\text{ref}}(p) + g_{ii}(p)\,f_i(p)\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} (-1)^{i+n} A_{in}(p) \left[K_{i\,\text{tot}}Y_{i\,\text{ref}}(p) + g_{ii}(p)\,f_i(p)\right] \end{Vmatrix} + \right. \\ & + \begin{Vmatrix} \sum_{i=1}^{n} \left[(-1)^{i+1} A_{i1}(p) \sum_{k=1}^{n} b_{ik}(p)\,f_k(p)\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} \left[(-1)^{i+j} A_{ij}(p) \sum_{k=1}^{n} b_{ik}(p)\,f_k(p)\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} \left[(-1)^{n+i} A_{in}(p) \sum_{k=1}^{n} b_{ik}(p)\,f_k(p)\right] \end{Vmatrix} + \\ & \left. + \begin{Vmatrix} \sum_{i=1}^{n} \left[(-1)^{i+1} A_{i1}(p) \sum_{k=2}^{n} c_{ik}(p)\,Y_{k\,\text{ref}}(p)\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} \left[(-1)^{i+j} A_{ij}(p) \sum_{\substack{k=1 \\ k\neq j}}^{n} c_{ik}(p)\,Y_{k\,\text{ref}}(p)\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} \left[(-1)^{i+n} A_{in}(p) \sum_{k=1}^{n-1} c_{ik}(p)\,Y_{k\,\text{ref}}(p)\right] \end{Vmatrix} \right\}. \end{aligned} \tag{2.33}$$

The matrix equation (2.33) can be partitioned to n equations in n controlled variables. These are obtained by equating the corresponding rows of the matrices in the right- and the left-hand sides of (2.33). The equation of any j-th controlled variable thus takes the form

$$Y_j = \frac{1}{\Delta}\left\{\sum_{i=1}^{n}(-1)^{i+j}A_{ij}(p)[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)+g_{ii}(p)f_i(p)]+\right.$$
$$+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(p)\sum_{k=1}^{n}b_{ik}(p)f_k(p)\right]+$$
$$\left.+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(p)\sum_{\substack{k=1\\k\neq j}}^{n}c_{ik}(p)Y_{k\,\mathrm{ref}}(p)\right]\right\}. \qquad (2.34)$$

Equation (2.34) is the most general expression for the j-th controlled variable in a system where the variables are coupled through the plant, the loads, and the measuring devices, but each variable is regulated by a single-loop subsystem. Let us consider some particular cases of this general equation.

(a) Ordinary multivariable control systems, where the coupling between the controlled variables is conditioned by the plant only. The equation of the j-th controlled variable in this case is easily obtained from (2.34) putting $b_{ik}(p)=c_{ik}(p)=0$:

$$Y_{j0}=\frac{1}{\Delta_0}\sum_{i=1}^{n}(-1)^{i+j}A_{0ij}(p)[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)+g_{ii}(p)f_i(p)]. \qquad (2.35)$$

Here Δ_0 and A_{0ij} are obtained from Δ and A on substituting $r_{ik}=0$.

(b) Multidimensional servosystems. The equation for the j-th controlled variable of one of the servos in a multidimensional servosystem is obtained from (2.34) by putting $b_{ik}(p)=0$ and $\alpha_{ik}(p)=0$:

$$Y_{j\mathrm{m}}=\frac{1}{\Delta_{\mathrm{m}}}\left\{\sum_{i=1}^{n}(-1)^{i+j}A_{\mathrm{m}ij}(p)[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)+g_{ii}(p)f_i(p)]+\right.$$
$$\left.+\sum_{i=1}^{n}(-1)^{i+j}A_{\mathrm{m}ij}(p)\sum_{\substack{k=1\\k\neq j}}^{n}c_{ik}Y_{k\,\mathrm{ref}}\right\}, \qquad (2.36)$$

where

$$\Delta_{\mathrm{m}}=\begin{Vmatrix} a_{11} & c_{12} & \dots & c_{1n}\\ c_{21} & a_{22} & \dots & c_{2n}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ c_{n1} & \dots & \dots & a_{nn}\end{Vmatrix}, \qquad (2.37)$$

and $A_{\mathrm{m}ij}$ are the cofactors of the corresponding elements in the determinant (2.37)*.

(c) Ordinary combined control system. We have already stressed that if the operators $b_{ik}(p)$ are appropriately chosen, equation (2.34) can be made to represent the j-th controlled variable in a multivariable combined-control system. In an ordinary combined control system, load signals,

* The subscript $\mathrm{m}\,ij$ indicates that the cofactor pertains to the element ij of the matrix of the multidimensional servosystem.

as well as the monitored deviation, are used as controlling factors. The equation for a combined control system with a single controlled variable is obtained without difficulty by putting $i=1$, $A_{ij}=0$, and $c_{ik}=0$ in (2.34). If now the system comprises several control loops which are load-coupled in the sense that the loads of the different loops are employed to improve the quality of each subsystem, the general equation is obtained from (2.34) by the above-mentioned substitution:

$$Y_{jk}(p)=\frac{1}{\Delta_k}\left\{A_{jjk}(p)[K_{j\,\mathrm{tot}}Y_{j\,\mathrm{ref}}(p)+g_{jj}(p)f_j(p)]+\ A_{jjk}(p)\sum_{k=1}^{n}b_{jk}(p)f_k(p)\right\}, \quad (2.38)$$

where

$$\Delta_k=\begin{vmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & a_{nn} \end{vmatrix}.$$

Having considered the various equations of multivariable control systems, we now proceed to discuss their operating conditions.

§ 2.4. STEADY-STATE OPERATION

We will derive a matrix equation for steady-state operation and establish some general properties of multivariable control systems under steady-state conditions. Remember that for the time being we are dealing with multivariable systems with single-loop subsystems.

The steady-state equation can be obtained from (2.15) by putting $p=0$. In explicit form, the equation for any j-th controlled variable under steady-state conditions is written from (2.34) as

$$\begin{aligned} Y_j(0)=\frac{1}{\Delta_s}\Bigg\{&\sum_{i=1}^{n}(-1)^{i+j}A_{ij}(0)[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(0)+g_{ii}(0)f_i(0)]+ \\ &+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(0)\sum_{k=1}^{n}b_{ik}(0)f_k(0)\right]+ \\ &+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(0)\sum_{\substack{k=1\\k\neq j}}^{n}c_{ik}(0)Y_{k\,\mathrm{ref}}(0)\right]\Bigg\}. \end{aligned} \quad (2.39)$$

It is readily seen that delay elements, if present, do not influence the steady-state operation of the system, since $\lim_{p\to 0} e^{\tau p}=1$.

Let m out of the total n control loops be integral, while the remaining $n-m$ loops are proportional. A single-loop system is called integral if and only if it contains at least one integrating (floating) controller /4, 5/. In proportional systems, the controller contains no integrating (floating) elements.

Let the elements be enumerated in such a way that the first m subscripts refer to integral subsystems. Then

$$\left.\begin{aligned}
&\lim_{p\to 0} a_{kk}(p) = K_{k\,\mathrm{tot}} && (k=1,\ 2,\ \ldots,\ m),\\
&\lim_{p\to 0} a_{kk}(p) = 1 + K_{\mathrm{tot}} && (k=m+1,\ \ldots,\ n),\\
&\lim_{p\to 0} a_{ki}(p) = K_{k\,\mathrm{tot}} r_{ki} && (k=1,\ 2,\ \ldots,\ m),\\
&\lim_{p\to 0} a_{ki}(p) = K_k a_{ki}(0) + K_{k\,\mathrm{tot}} r_{ki} && (k=m+1,\ \ldots,\ n),\\
&\lim_{p\to 0} g_{kk}(p) = 0 && (k=1,\ 2,\ \ldots,\ m),\\
&\lim_{p\to 0} g_{kk}(p) = K_k && (k=m+1,\ \ldots,\ n),\\
&\lim_{p\to 0} b_{ki}(p) = 0 && (k=1,\ \ldots,\ m),\\
&\lim_{p\to 0} b_{ki}(p) = K_k \beta_{ki}(0) && (k=m+1,\ \ldots,\ n).
\end{aligned}\right\} \tag{2.40}$$

We now proceed to determine the j-th controlled variable in two limiting cases: all the subsystems are integral (case 1), or they are all proportional (case 2). In case 1 we have $m=n$, and in case 2 $m=0$.

In case 1, the various elements in equation (2.39) are written in explicit form

$$\left.\begin{aligned}
&\Delta_1 = \begin{vmatrix} K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}} r_{12} & \ldots & K_{1\,\mathrm{tot}} r_{1n} \\ K_{2\,\mathrm{tot}} r_{21} & K_{2\,\mathrm{tot}} & \ldots & K_{2\,\mathrm{tot}} r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n\,\mathrm{tot}} r_{n1} & \ldots & \ldots & K_{n\,\mathrm{tot}} \end{vmatrix},\\
&g_{kk}(0) = 0, \quad b_{ki}(0) = 0,\\
&a_{ki}(0) = K_{k\,\mathrm{tot}} r_{ki}.
\end{aligned}\right\} \tag{2.41}$$

The transpose in this case is

$$A_t(0) = \begin{Vmatrix} K_{1\,\mathrm{tot}} & K_{2\,\mathrm{tot}} r_{21} & \ldots & K_{n\,\mathrm{tot}} r_{n1} \\ K_{1\,\mathrm{tot}} r_{12} & K_{2\,\mathrm{tot}} & \ldots & K_{n\,\mathrm{tot}} r_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ K_{1\,\mathrm{tot}} r_{1n} & K_{2\,\mathrm{tot}} r_{21} & \ldots & K_{n\,\mathrm{tot}} \end{Vmatrix}. \tag{2.42}$$

Inserting (2.41) in (2.39) and making use of (2.42), we find

$$Y_j(0) = \frac{1}{\Delta_1} \sum_{i=1}^{n} (-1)^{i+j} A_{ij}(0) \left[K_{i\,\mathrm{tot}} Y_{i\,\mathrm{ref}} + \sum_{\substack{k=1 \\ k\neq j}}^{n} c_{ik}(0)\, Y_{\mathrm{ref}\,k} \right]. \tag{2.43}$$

If the measuring elements are uncoupled, we have

$$A_t = A_0 = \begin{Vmatrix} K_{1\,\mathrm{tot}} & 0 & 0 \ldots 0 \\ 0 & K_{2\,\mathrm{tot}} & 0 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots K_{n\,\mathrm{tot}} \end{Vmatrix},$$

$$Y_{j\,1}(0) = \begin{vmatrix} K_{1\,\mathrm{tot}} & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & K_{j-1\,\mathrm{tot}} & 0 & 0 \\ 0 & \ldots & K_{j+1\,\mathrm{tot}} & 0 \\ \cdots & \cdots & \cdots & K_{n\,\mathrm{tot}} \end{vmatrix} \Big/ \begin{vmatrix} K_{1\,\mathrm{tot}} & 0 & \ldots & 0 \\ 0 & K_{2\,\mathrm{tot}} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & K_{n\,\mathrm{tot}} \end{vmatrix} K_{j\,\mathrm{tot}} Y_{\mathrm{ref}}(0) = Y_{j\,\mathrm{ref}}(0).$$

We thus arrive at the following remarkable conclusion: if the subsystems are all integral, the steady-state value of a given controlled variable is independent of its own load and of the load of the other subsystems; it is furthermore independent of the other controlled variables (although without control, all the variables are plant- and load-coupled), depending only on the artificially introduced coupling coefficients between the measuring elements, which in a sense alter only the reference value. If the measuring elements are uncoupled, the controlled variable is equal to its reference value. This result can be alternatively stated as follows: if all the subsystems are integral, the controlled variables are independent in the steady-state and the system is said to be statically noninteracting*.

If, however, the measuring devices are coupled, the steady-state value of each controlled variable is dependent not only on its own reference value but also on the reference values of all the other variables.

Let us now consider the case of proportional subsystems, assuming that the controlled object does not contain integrating elements either. The system determinant is written as

$$\Delta_s = \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12} & \dots & K_1\alpha_{1n}(0)+K_{1\,\mathrm{tot}}r_{1n} \\ \dots & \dots & \dots & \dots \\ K_j\alpha_{j1}(0)+K_{j\,\mathrm{tot}}r_{j1} & \dots & & K_j\alpha_{jn}(0)+K_{j\,\mathrm{tot}}r_{jn} \\ \dots & \dots & \dots & \dots \\ K_n\alpha_{n1}(0)+K_{n\,\mathrm{tot}}r_{n1} & \dots & \dots & 1+K_{n\,\mathrm{tot}} \end{vmatrix}. \qquad (2.44)$$

The transpose A_{ts} under steady-state conditions is

$$A_{ts} = \begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & K_j\alpha_{j1}(0)+K_{j\,\mathrm{tot}}r_{j1} & \dots & K_n\alpha_{n1}(0)+K_{n\,\mathrm{tot}}r_{n1} \\ K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12} & 1+K_{j\,\mathrm{tot}} & \dots & K_n\alpha_{nj}(0)+K_{n\,\mathrm{tot}}r_{nj} \\ \dots & \dots & \dots & \dots \\ K_1\alpha_{1n}(0)+K_{1\,\mathrm{tot}}r_{1n} & K_j\alpha_{jn}(0)+K_{j\,\mathrm{tot}}r_{jn} & \dots & 1+K_{n\,\mathrm{tot}} \end{Vmatrix}. \qquad (2.45)$$

Equation (2.39) is thus rewritten as

$$Y_{j\,s}(0) = \frac{1}{\Delta_s}\sum_{i=1}^{n}(-1)^{i+j}A_{ij}(0)\left\{K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(0)+g_{ii}(0)f(0)+\right.$$
$$\left.+\sum_{k=1}^{n}b_{ik}(0)f_k(0)+\sum_{\substack{k=1\\k\neq j}}^{n}c_{ik}Y_{k\,\mathrm{ref}}(0)\right\}. \qquad (2.46)$$

Each A_{ij} is the determinant (2.44) with one row and one column crossed out. The degree of the determinant in the numerator of (2.46) is thus always one less than the degree of the determinant Δ_s.

It is easily seen that as the controller gain K increases indefinitely, we have

$$\lim_{K_{j\,\mathrm{tot}}\to\infty} Y_{j\,s}(0) = Y_{j\,\mathrm{ref}}(0)+\sum_{\substack{k=1\\k\neq j}}^{n} r_{ik}Y_{k\,\mathrm{ref}}(0). \qquad (2.47)$$

This increase in gain is of course permissible only if the system retains its stability.

Thus, if the gain $K_{i\,\mathrm{tot}}$ of each control loop is increased by increasing the corresponding controller gain, each controlled variable in the limit

* The general case of noninteracting (autonomous) systems is treated in a special section of Chapter Six.

is equal to its reference value, appropriately modified by introduction of artificial coupling between the measuring elements; it is thus independent of the other controlled variables and loads. If the measuring elements are uncoupled, we have

$$\lim_{K_{j\,\mathrm{tot}}\to\infty} Y_{j\,s}(0) = Y_{j\,\mathrm{ref}}(0). \tag{2.48}$$

From (2.46) and the values of the elements entering equation (2.46) we see that if the coefficients $K_{j\,\mathrm{tot}}$ are finite, the individual controlled variables are coupled, but the interdependence diminishes as the gain factors of the individual controllers increase.

If all the plant and controller parameters are known, equation (2.46) can be applied to determine the steady-state value of the controlled variable and hence to establish the relationship between the controlled variable and the load.

As an example, we calculate the steady-state value of, say, the second controlled variable in a three-variable system:

$$Y_{2s}(0) = \frac{1}{\Delta_s}\sum_{l=1}^{3}(-1)^{l+2}A_{l2}(0)\left\{K_{l\,\mathrm{tot}}Y_{l\,\mathrm{ref}}(0) + \right.$$
$$\left. + g_{ll}(0)f_l(0) + \sum_{k=1}^{3} b_{lk}(0)f_n + \sum_{\substack{k=1\\k\neq 2}}^{3} c_{lk}Y_{k\,\mathrm{ref}}(0)\right\}. \tag{2.49}$$

From (2.40) we have

$$g_{ll}(0) = K_l, \quad b_{ik} = K_i\beta_{lk}(0), \quad c_{ik} = K_{i\,o6}r_{lk},$$

$$\Delta_s = \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12}(0) & K_1\alpha_{13}(0)+K_{1\,\mathrm{tot}}r_{13}(0) \\ K_2\alpha_{21}(0)+K_{2\,\mathrm{tot}}r_{21}(0) & 1+K_{2\,\mathrm{tot}} & K_2\alpha_{23}(0)+K_{2\,\mathrm{tot}}r_{23}(0) \\ K_3\alpha_{31}(0)+K_{3\,\mathrm{tot}}r_{31}(0) & K_3\alpha_{32}(0)+K_{3\,\mathrm{tot}}r_{32}(0) & 1+K_{3\,\mathrm{tot}} \end{vmatrix} \tag{2.50}$$

$$\left.\begin{aligned} A_l &= \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_2\alpha_2(0)+K_{2\,\mathrm{tot}}r_{21}(0) & K_3\alpha_{31}(0)+K_{3\,\mathrm{tot}}r_{31}(0) \\ K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12}(0) & 1+K_{2\,\mathrm{tot}} & K_3\alpha_{32}(0)+K_{3\,\mathrm{tot}}r_{32}(0) \\ K_1\alpha_{13}(0)+K_{1\,\mathrm{tot}}r_{13}(0) & K_2\alpha_{23}(0)+K_{2\,\mathrm{tot}}r_{23}(0) & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \\ A_{12} &= -\begin{vmatrix} K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12}(0) & K_3\alpha_{32}(0)+K_{3\,\mathrm{tot}}r_{32}(0) \\ K_1\alpha_{13}(0)+K_{1\,\mathrm{tot}}r_{12}(0) & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \\ A_{22} &= \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_3\alpha_{31}(0)+K_{3\,\mathrm{tot}}r_{31}(0) \\ K_1\alpha_{13}(0)+K_{1\,\mathrm{tot}}r_{13}(0) & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \\ A_{32} &= -\begin{vmatrix} K_1\alpha_{12}(0)+K_{1\,\mathrm{tot}}r_{12}(0) & 1+K_{2\,\mathrm{tot}} \\ K_1\alpha_{13}(0)+K_{1\,\mathrm{tot}}r_{13}(0) & K_2\alpha_{23}(0)+K_{2\,\mathrm{tot}}r_{23}(0) \end{vmatrix}. \end{aligned}\right\} \tag{2.51}$$

Inserting the appropriate numerical values, we obtain Y_{2s}.

From (2.49) we see that, by introducing additional load coupling, we may achieve any desired variation of the steady-state controlled variable as a function of load. Note that the number of disturbances or loads need not be equal to the number of controlled variables; furthermore, introduction of a certain number of disturbing factors in addition to the already existing disturbances in the system does not involve any fundamental difficulties.

Let us now consider the general case, when some of the subsystems are integral and the others are proportional. In our example of a three-variable system, we assume that Y_2 is under integral control. The coupling between the measuring elements is ignored, since it is artificially introduced into the system and only alters the reference value of the controlled variable.

We thus operate under the following conditions:

$$\left.\begin{array}{c} r_{ik}(0)=0, \quad a_{21}(0)=0, \quad g_{22}(0)=0, \\ a_{22}(0)=K_{2\text{tot}}, \quad b_{2l}(0)=0. \end{array}\right\} \tag{2.52}$$

Making use of (2.52), we write

$$Y_{2s}(0)=\frac{K_{2\,\text{tot}}Y_{2,\text{ref}}(0)\begin{vmatrix} 1+K_{1\,\text{tot}} & K_1\alpha_{13}(0) \\ K_3\alpha_{31}(0) & 1+K_{3\,\text{tot}} \end{vmatrix}}{\begin{vmatrix} 1+K_{1\text{tot}} & 0 & K_1\alpha_{13}(0) \\ 0 & K_{2\,\text{tot}} & 0 \\ K_3\alpha_{31}(0) & 0 & 1+K_{3\text{tot}} \end{vmatrix}}=Y_{2\,\text{ref}}(0). \tag{2.53}$$

Thus, under steady-state conditions, the integral variables are load-independent and do not interact with other controlled variables, despite the plant-coupling. The only case when disturbances may alter the integral controlled variable is if they are mixed with the reference value; however, in steady-state conditions the additional signal causes an equivalent change in the reference signal.

We now establish the interaction of integral variables with proportional variables. Suppose that in our three-variable case, the second and third variables are integral, while the first variable is proportional. The equation of the first controlled variable (again ignoring the transducer coupling) according to (2.49) is

$$Y_{1s}(0)=\frac{1}{\Delta_s}\sum_{l=1}^{3}(-1)^{l+1}A_{l1}(0)\left\{K_{l\,\text{tot}}Y_{l\text{ref}}(0)+g_{l1}(0)f_1(0)+\sum_{k=1}^{3}b_{lk}(0)f_k(0)\right\}. \tag{2.54}$$

Substituting for the elements in (2.54), we find

$$\begin{aligned} Y_{1s}(0)=&\frac{1}{\begin{vmatrix} 1+K_{1\,\text{tot}} & K_1\alpha_{12}(0) & K_1\alpha_{13}(0) \\ 0 & K_{2\,\text{tot}} & 0 \\ 0 & 0 & K_{3\text{tot}} \end{vmatrix}}\left\{\begin{vmatrix} K_{2\,\text{tot}} & 0 \\ 0 & K_{3\text{tot}} \end{vmatrix}[K_{1\text{tot}}Y_{1\text{ref}}+K_1f_1(0)+\right. \\ &+K_1(\beta_{11}f_1(0)+\beta_{12}f_2(0)+\beta_{13}f_3(0))]- \\ &\left.-\begin{vmatrix} K_1\alpha_{12}(0) & 0 \\ K_1\alpha_{13}(0) & K_{3\,\text{tot}} \end{vmatrix}[K_{2\,\text{tot}}Y_{2\text{ref}}]+\begin{vmatrix} K_1\alpha_{12}(0) & K_{2\text{tot}} \\ K_1\alpha_{13}(0) & 0 \end{vmatrix}[K_{3\text{tot}}Y_{3\text{ref}}]\right\}. \end{aligned} \tag{2.55}$$

After simple manipulations, we obtain

$$\begin{aligned} Y_{1s}=&\frac{K_{1\,\text{tot}}}{1+K_{1\text{tot}}}Y_{1\,\text{ref}}(0)+\frac{K_1f_1(0)}{1+K_{1\,\text{tot}}}+\frac{K_1}{1+K_{1\,\text{tot}}}[\beta_{11}f_1(0)+\beta_{12}f_2(0)+ \\ &+\beta_{13}f_3(0)]-\frac{K_1}{1+K_{1\text{tot}}}[\alpha_{12}(0)Y_{2\text{ref}}+\alpha_{13}Y_{3\text{ref}}]. \end{aligned} \tag{2.55a}$$

The physical meaning of the components in equation (2.55a) is obvious: the first term in the right-hand side corresponds to proportional control of the given variable, when considered separately, the second and third terms represent the effect of the variable's own load and of the additional load of this and other variables introduced through the transducer β_{ik}; the last term describes the effect of the extraneous reference values on the steady-state value of the controlled variable. From equation (2.55a) it is also easily seen that the effect of the other controlled variables and their loads in the steady-state conditions increases with the increase in plant gain and decreases with the increase in the gain parameter of the controller or the proportional control loop.

In conclusion of this section, we consider the case when the transfer functions of the plant and the controllers are identically equal for all the variables. We shall try to establish the behavior of this system under steady-state conditions. Since all the subsystems are identical, they are all either proportional or integral. The case of integral subsystems is of no significance for our analysis, since as we have shown in the preceding for a more general case, the subsystems are independent under steady-state conditions.

We thus consider the case of proportional subsystems, remembering that subsystem transfer functions and the coupling coefficients determined by the plant properties are respectively equal to one another. In this case the matrix A is equal to its transpose A_t and is symmetrical: $\alpha_{ik}(p) = \alpha_{ki}(p)$, where i and k are subscripts pertaining to any controlled variable; $a_{ii}(p) = a_{kk}(p)$, $a_{ij}(p) = a_{kn}(p)$.

Let us consider the case of an ordinary multivariable control system, ignoring load- and transducer-coupling.

The matrix A in this case is

$$A_d = A_{dt} = \begin{Vmatrix} a_{11}, a & a & \dots & a \\ a & a_{22}, a & \dots & a \\ a & a & \dots & a \\ \cdot & \cdot & \cdot & \cdot \\ a & a & \dots & a, a_{nn} \end{Vmatrix}. \qquad (2.56)$$

The cofactors of all the diagonal elements in (2.56) are obviously equal, i.e., $A_{ii} = A_{jj}$, and they all have the sign plus. We can also prove the following proposition:

The cofactors of all the other elements in the matrix (2.56) are also equal to one another, but have the sign minus.

Indeed, since all the nondiagonal elements of the matrix (2.56) are equal to one another and the diagonal elements are also equal to one another, the cofactors of any two adjoining nondiagonal elements will coincide if the corresponding pair of rows and columns is interchanged in one of the cofactors. This operation, however, will reverse the sign of the cofactor, but since the cofactors of two adjoining elements have different signs, it is clear that in virtue of symmetry the cofactors are equal in magnitude and in sign. This proves the first half of the proposition.

We will now show that all the cofactors reduced to identical form have the sign minus. It suffices to show that at least one of the cofactors has the sign minus. Consider the cofactor of an element adjoining a diagonal element. Since the cofactor of a diagonal element always has the sign plus, the cofactor of an adjoining nondiagonal element must inevitably have the sign minus, which completes the proof.

Making use of the above conditions and the symmetry of the matrix, we obtain from the general equation (2.34) the following expression of the j-th controlled variable:

$$Y_j = \frac{1}{\Delta_d}[A_{jj}(p)[K_{j\text{tot}}Y_{j\text{ref}}(p) + g_{jj}(p)f_j(p)] - (n-1)A_{j+1}[K_{j+1\text{tot}}Y_{j+1\text{ref}}(p) + g_{jj}(p)f_j(p)]]. \qquad (2.57)$$

Since all the reference values are equal, equation (2.57) can be written as

$$Y_j = [K_{j\,\mathrm{tot}} Y_{j\,\mathrm{ref}} + g_{jj}(p) f_j(p)] \frac{A_{jj}(p) - (n-1) A_{j,\,j+1}(p)}{\Delta_d} =$$

$$= [K_{j\,\mathrm{tot}} Y_{j\,\mathrm{ref}} + g_{jj}(p) f_j(p)] \frac{A_{jj}(p) - (n-1) A_{j,\,j+1}(p)}{a_{jj}(p) A_{jj}(p) - (n-1) a_{j,\,j+1}(p) A_{j,\,j+1}(p)}. \qquad (2.58)$$

The analysis of the system is considerably simplified in this case. Indeed, the stability of the entire system is determined by the position of the roots of the denominator in (2.58)

$$a_{jj}(p) A_{jj}(p) - (n-1) a_{j,\,j+1}(p) A_{j,\,j+1}(p) = 0 \qquad (2.58a)$$

relative to the imaginary axis. It is easily seen that equation (2.58a) can be reduced to the following form:*

$$[a_{jj}(p) - a_{j,\,j+1}(p)]^{n-1} [a_{jj} + (n-1) a_{j,\,j+1}(p)] = 0.$$

It thus suffices to investigate two equations of a much simpler form, namely

$$a_{jj}(p) - a_{j,\,j+1}(p) = 0$$

and

$$a_{jj} + (n-1) a_{j,\,j+1}(p) = 0.$$

This approach to stability is very attractive, since the order of the equations to be investigated is equal to the order of the subsystem. It should however be kept in mind that the results should further be tested for coarseness in the sense of A. A. Andronov. This test is particularly important in our case, since the smallest deviation from homogeneity will markedly increase the order of the equation to be investigated for stability.

Under steady-state conditions, equation (2.58) takes the form

$$Y_{js}(0) = [K_{j\,\mathrm{tot}} Y_{j\,\mathrm{ref}}(0) + g_{jj}(0) f(0)] \frac{A_{jj}(0) - (n-1) A_{j+1}(0)}{(1 + K_{j\,\mathrm{tot}}) A_{jj}(0) - (n-1) K_{j\alpha}(0) A_{j\,j+1}(0)}. \qquad (2.59)$$

§ 2.5. ERRORS IN MULTIVARIABLE CONTROL SYSTEMS WITH BASIC ELEMENTS

We resume our discussion of multivariable control systems with subsystems made up of basic elements in single-loop configuration.

We introduce the concept of an error matrix in the general case of a multivariable control system. The definition is analogous to that proposed for multidimensional servosystems /80/. The elements of the error matrix X are defined as $X_i = Y_{i\,\mathrm{ref}} - Y_i$. Eliminating Y_i and Y_i' between

* This result is due to A. A. Krasovskii /23/.

(2.7) and (2.10) and seeing that $Y_i = Y_{i\,\mathrm{ref}} - X_i$, we obtain the following expressions for X_i:

$$\begin{aligned}
&\left[D_i(p)\,Q_i(p)\,R_i(p)\,e^{\tau_i p} + K_i K_{i_a}\delta_i\mu_i\right] X_i(p) + \\
&+\left[K_i\mu_i Q_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p) - \mu_i D_i(p)\,Q_i(p)\,e^{\tau_i p}\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)\right] X_k(p) = \\
&= \mu_i D_i(p)\,Q_i(p)\,e^{\tau_i p} Y_{i\,\mathrm{ref}}(p) + K_i\mu_i Q_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\,Y_{k\,\mathrm{ref}}(p) + \\
&\qquad + K_i\mu_i Q_i(p)\,f_i(p) \qquad (i = 1,\ 2,\ \ldots,\ n). \qquad (2.60)
\end{aligned}$$

Our aim is to write equation (2.60) in matrix form. We put

$$\left.\begin{aligned}
&D_i(p)\,Q_i(p)\,R_i(p)\,e^{\tau_i p} + K_i K_{i_a}\delta_i\mu_i = a_{ii}(p),\\
&\mu_i D_i(p)\,Q_i(p)\,e^{\tau_i p} = \zeta_i(p),\\
&K_i\mu_i Q_i(p) = \gamma_i(p).
\end{aligned}\right\} \qquad (2.61)$$

In this notation, equations (2.60) can be written in the matrix form as follows:

$$AX = BY_{\mathrm{ref}} + CF, \qquad (2.62)$$

where

$$A = \begin{Vmatrix}
a_{11}(p) & \gamma_1(p)\,\alpha_{12}(p) - \zeta_1(p)\,r_{12}(p) & \ldots & \gamma_1(p)\,\alpha_{1n}(p) - \zeta_1(p)\,r_{1n}(p)\\
\gamma_2(p)\,\alpha_{21}(p) - \zeta_2(p)\,r_{21}(p) & a_{22}(p) & \ldots & \gamma_2(p)\,\alpha_{2n}(p) - \zeta_2(p)\,r_{2n}(p)\\
\ldots & \ldots & \ldots & \ldots\\
\gamma_n(p)\,\alpha_{n1}(p) - \zeta_n(p)\,r_{n1}(p) & & \ldots & a_{nn}(p)
\end{Vmatrix}, \qquad (2.63)$$

$$B = \begin{Vmatrix}
\zeta_1(p) & \gamma_1(p)\,\alpha_{12}(p) & \ldots & \gamma_1(p)\,\alpha_{1n}(p)\\
\gamma_2(p)\,\alpha_{21}(p) & \zeta_2(p) & \ldots & \gamma_2(p)\,\alpha_{2n}(p)\\
\ldots & \ldots & \ldots & \ldots\\
\gamma_n(p)\,\alpha_{n1}(p) & \gamma_n(p)\,\alpha_{n2}(p) & \ldots & \zeta_3(p)
\end{Vmatrix}, \qquad (2.64)$$

$$C = \begin{Vmatrix}
\gamma_1(p) & 0 & \ldots & 0\\
0 & \gamma_2(p) & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots\\
0 & 0 & \ldots & \gamma_n(p)
\end{Vmatrix}, \qquad (2.65)$$

$$X = \begin{Vmatrix} X_1\\ X_2\\ \vdots\\ X_n \end{Vmatrix}, \quad Y_{\mathrm{ref}} = \begin{Vmatrix} Y_{1\,\mathrm{ref}}(p)\\ Y_{2\,\mathrm{ref}}(p)\\ \vdots\\ Y_{n\,\mathrm{ref}}(p) \end{Vmatrix}, \qquad (2.66)$$

$$F = \begin{Vmatrix} f_1(p)\\ f_2(p)\\ \vdots\\ f_n(p) \end{Vmatrix}. \qquad (2.67)$$

From (2.62) we obtain the error matrix

$$X = A^{-1}[BY_{\mathrm{ref}} + CF]. \qquad (2.68)$$

The matrix A can be written as a product of two matrices

$$A=\begin{Vmatrix} a_{11}(p) & 0 & \dots & 0 \\ 0 & a_{22}(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(p) \end{Vmatrix} \times \begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ R_{21} & 1 & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & \dots & \dots & 1 \end{Vmatrix}, \tag{2.69}$$

where

$$R_{12}=\frac{\gamma_1(p)\alpha_{12}(p)-\zeta_1(p)r_{12}(p)}{a_{11}(p)},$$
$$R_{1n}=\frac{\gamma_1(p)\alpha_{1n}(p)-\zeta_1(p)r_{1n}(p)}{a_{11}(p)},$$
$$R_{21}=\frac{\gamma_2(p)\alpha_{21}(p)-\zeta_2(p)r_{21}(p)}{a_{22}(p)},$$
$$R_{2n}=\frac{\gamma_2(p)\alpha_{2n}(p)-\zeta_2(p)r_{2n}(p)}{a_{22}(p)},$$
$$R_{n1}=\frac{\gamma_n(p)\alpha_{n1}(p)-\zeta_n(p)r_{n1}(p)}{a_{nn}(p)},$$

are likewise for the matrix B

$$B=\begin{Vmatrix} 1 & \frac{\gamma_1(p)\alpha_{12}(p)}{\zeta_2(p)} & \dots & \frac{\gamma_1(p)\alpha_{1n}(p)}{\zeta_n(p)} \\ \frac{\gamma_2(p)\alpha_{21}(p)}{\zeta_1(p)} & 1 & \dots & \frac{\gamma_2(p)\alpha_{2n}(p)}{\zeta_n(p)} \\ \dots & \dots & \dots & \dots \\ \frac{\gamma_n(p)a_{n1}(p)}{\zeta_1(p)} & \dots & \dots & 1 \end{Vmatrix} \times \begin{Vmatrix} \zeta_1(p) & 0 & \dots & 0 \\ 0 & \zeta_2(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \zeta_n(p) \end{Vmatrix}. \tag{2.70}$$

Making use of (2.59), we write the inverse A^{-1} in the form

$$A^{-1}=\begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ R_{21} & 1 & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & \dots & \dots & 1 \end{Vmatrix}^{-1} \times \begin{Vmatrix} \frac{1}{a_{11}(p)} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}(p)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(p)} \end{Vmatrix}. \tag{2.71}$$

Substituting (2.70) and (2.71) in (2.68), we obtain

$$x=\begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & \dots & \dots & 1 \end{Vmatrix}^{-1} \times \begin{Vmatrix} \frac{1}{a_{11}(p)} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}(p)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(p)} \end{Vmatrix} \times$$
$$\times \begin{Vmatrix} 1 & \frac{\gamma_1(p)\alpha_{12}(p)}{\zeta_2(p)} & \dots & \frac{\gamma_1(p)\alpha_{1n}(p)}{\zeta_n(p)} \\ \frac{\gamma_2(p)\alpha_{21}(p)}{\zeta_1(p)} & 1 & \dots & \frac{\gamma_2(p)\alpha_{2n}(p)}{\zeta_n(p)} \\ \dots & \dots & \dots & \dots \\ \frac{\gamma_n(p)\alpha_{n1}(p)}{\zeta_1(p)} & \dots & \dots & 1 \end{Vmatrix} \times$$
$$\times \begin{Vmatrix} \zeta_1(p) & 0 & \dots & 0 \\ 0 & \zeta_2(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \zeta_n(p) \end{Vmatrix} \times \begin{Vmatrix} Y_{1\,\mathrm{ref}} \\ Y_{2\,\mathrm{ref}} \\ \vdots \\ Y_{n\,\mathrm{ref}} \end{Vmatrix} + \begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ R_{21} & 1 & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & \dots & \dots & 1 \end{Vmatrix}^{-1} \times$$
$$\times \begin{Vmatrix} \frac{\gamma_1(p)}{a_{11}(p)} & 0 & \dots & 0 \\ 0 & \frac{\gamma_2(p)}{a_{22}(p)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\gamma_n(p)}{a_{nn}(p)} \end{Vmatrix} \times \begin{Vmatrix} f_1(p) \\ f_2(p) \\ \vdots \\ f_n(p) \end{Vmatrix}. \tag{2.72}$$

Consider the first term in (2.72). It determines the dynamic properties of a multivariable system without load and of systems where the transient process is initiated by a disturbance at all the subsystem inputs or, equivalently, by application of the reference values $Y_{i\,\mathrm{ref}}(p)$ to the system inputs.

Consider an isolated, noninteracting system. Its transfer function can be obtained by putting $\alpha_{ik}(p) = r_{ik}(p) = 0$, and in our particular case $f_i(p) = 0$. Thus

$$X_i(p) = \frac{\zeta_i(p)}{a_{ii}(p)} Y_{i\,\mathrm{ref}}(p). \tag{2.73}$$

We will now determine the system error. From the properties of Laplace transformation we know that

$$\lim_{t\to\infty} x(t) = \lim_{p\to 0} p \frac{\zeta_i(p)}{a_{ii}(p)} Y_{i\,\mathrm{ref}}(p). \tag{2.74}$$

Let $y_{i\,\mathrm{ref}}(t)$ be a step function, then

$$y_{i\,\mathrm{ref}} = \frac{Y_{i\,\mathrm{ref}}(0)}{p}.$$

Thus

$$\lim_{t\to\infty} x(t) = \frac{\zeta_i(0)}{a_{ii}(0)} Y_{i\,\mathrm{ref}}(0).$$

Here $\frac{\zeta_i(0)}{a_{ii}(0)}$ is the proportional or the zeroth error. If the system is integral to a certain degree, errors of higher order can be obtained.

Let the respective errors of isolated, noninteracting systems be $K_1^{(S_1)}, K_2^{(S_2)}, \ldots, K_n^{(S_n)}$, where the subscript identifies the system and the superscript is the order of the system error*.

We now return to the first term in (2.72) and postmultiply it by the identity matrix

$$E = \bar{a}a^{-1}, \tag{2.75}$$

where

$$a^{-1} = \begin{Vmatrix} \frac{1}{a_{11}(p)} & 0 & \ldots & 0 \\ & \frac{1}{a_{22}(p)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{1}{a_{nn}(p)} \end{Vmatrix}. \tag{2.76}$$

Making use of the peculiar property of the inverse of a diagonal matrix, we write the first term from (2.72) in the form

$$X = \begin{Vmatrix} 1 & \ldots & \gamma_1(p)\alpha_{1n}(p) - \frac{\zeta_1(p) r_{1n}(p)}{a_{11}(p)} \\ \ldots & \ldots & \ldots \\ \gamma_n(p)\alpha_{1n}(p) - \frac{\zeta_n(p) r_{n1}(p)}{a_{nn}(p)} & \ldots & 1 \end{Vmatrix}^{-1} \times$$

* The order of the error is determined by the degree of integral action of the system.

$$\times\begin{Vmatrix} \frac{1}{a_{11}(p)} & 0 & \dots & 0 \\ & \frac{1}{a_{22}(p)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(p)} \end{Vmatrix} \times \begin{Vmatrix} 1 & \frac{\gamma_1(p)\alpha_{12}(p)}{\zeta_2(p)} & \dots & \frac{\gamma_1(p)\alpha_{1n}(p)}{\zeta_n(p)} \\ \frac{\gamma_2(p)\alpha_{21}(p)}{\zeta_1(p)} & 1 & \dots & \frac{\gamma_2(p)\alpha_{2n}(p)}{\zeta_n(p)} \\ \dots & \dots & \dots & \dots \\ \frac{\gamma_n(p)\alpha_{n1}(p)}{\zeta_1(p)} & \dots & \dots & 1 \end{Vmatrix} \times$$

$$\times\begin{Vmatrix} a_{11}(p) & 0 & \dots & 0 \\ 0 & a_{22}(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(p) \end{Vmatrix} \times \begin{Vmatrix} Y_{1\,\mathrm{ref}} \\ \vdots \\ Y_{n\,\mathrm{ref}} \end{Vmatrix} \times \begin{Vmatrix} \frac{\zeta_1(p)}{a_{11}(p)} & 0 & \dots & 0 \\ 0 & \frac{\zeta_2(p)}{a_{22}(p)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{\zeta_n(p)}{a_{nn}(p)} \end{Vmatrix}. \qquad (2.77)$$

Under steady-state conditions, equation (2.77) can be written as

$$\lim_{t\to\infty} x(t) = \begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & 1 \end{Vmatrix}^{-1} \begin{Vmatrix} \frac{1}{a_{11}(0)} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}(0)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(0)} \end{Vmatrix} \times$$

$$\times\begin{Vmatrix} 1 & \frac{\gamma_1(0)\alpha_{12}(0)}{\zeta_2(0)} & \dots & \frac{\gamma_1(0)\alpha_{1n}(0)}{\zeta_n(0)} \\ \dots & \dots & \dots & \dots \\ \frac{\gamma_n(0)\alpha_{n1}(0)}{\zeta_1(0)} & \dots & \dots & 1 \end{Vmatrix} \times \begin{Vmatrix} a_{11}(0) & 0 & \dots & 0 \\ 0 & a_{22}(0) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(0) \end{Vmatrix} \times$$

$$\times\begin{Vmatrix} K_1^{(S_1)} & 0 & 0 \\ 0 & K_2^{(S_2)} & 0 \\ \dots & \dots & \dots \\ 0 & \dots & K_n^{(S_n)} \end{Vmatrix} \times \begin{Vmatrix} Y_{1\,\mathrm{ref}}(0) \\ \vdots \\ Y_{n\,\mathrm{ref}}(0) \end{Vmatrix}, \qquad (2.78)$$

where

$$R_{12} = \frac{\gamma_1(0)\alpha_{12}(0) - \zeta_1(0) r_{12}(0)}{a_{11}(0)},$$

. . . . . .

$$R_{1n} = \frac{\gamma_1(0)\alpha_{1n}(0) - \zeta_1(0) r_{1n}(0)}{a_{11}(0)},$$

. . . . . .

$$R_{n1} = \frac{\gamma_{n1}(0)\alpha_{n1}(0) - \zeta_n(0) r_{n1}(0)}{a_{nn}(0)}.$$

The expression

$$K = \begin{Vmatrix} 1 & R_{12} & \dots & R_{1n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & \dots & 1 \end{Vmatrix}^{-1} \times \begin{Vmatrix} \frac{1}{a_{11}(0)} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}(0)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}(0)} \end{Vmatrix} \times \begin{Vmatrix} 1 & \frac{\gamma_1(0)\alpha_{12}(0)}{\zeta_2(0)} & \dots & \frac{\gamma_1\alpha_{1n}(0)}{\zeta_n(0)} \\ \dots & \dots & \dots & \dots \\ \frac{\gamma_n(0)\alpha_{n1}(0)}{\zeta_1(0)} & \dots & \dots & 1 \end{Vmatrix} \times$$

$$\times\begin{Vmatrix} a_{11}(0) & 0 & \dots & 0 \\ 0 & a_{22}(0) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}(0) \end{Vmatrix} \times \begin{Vmatrix} K_1^{(S_1)} & 0 & 0 \\ 0 & K_2^{(S_2)} & 0 \\ \dots & \dots & \dots \\ 0 & 0 & K_n^{(S_n)} \end{Vmatrix} \qquad (2.79)$$

is the generalized error matrix. In the case

$$Q_i(0)=K_i\mu_i,\quad \zeta_i(0)=\mu_i,\quad \zeta_i=\mu_i \quad \text{and} \quad a_{ii}=1+K_{i\,\text{tot}},$$

the generalized error matrix takes the form

$$K=\left\|\begin{array}{ccc} 1 & \dots & \dfrac{K_1\mu_1\alpha_{1n}-\mu_1 r_{1n}}{1+K_{1\,\text{tot}}} \\ \dots & \dots & \dots \\ \dfrac{K_n\mu_n\alpha_{n1}-\mu_n r_{n1}}{1+K_{n\,\text{tot}}} & \dots & 1 \end{array}\right\|^{-1} \left\|\begin{array}{ccc} \dfrac{1}{1+K_{1\,\text{tot}}} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \dfrac{1}{1+K_{n\,\text{tot}}} \end{array}\right\| \times$$

$$\times\left\|\begin{array}{cccc} 1 & K_1\alpha_{12} & \dots & K_1\alpha_{1n} \\ K_2\alpha_{21} & 1 & \dots & K_2\alpha_{2n} \\ \dots & \dots & \dots & \dots \\ K_n\alpha_{n1} & K_n\alpha_{n2} & \dots & 1 \end{array}\right\| \times \left\|\begin{array}{cccc} 1+K_{1\,\text{tot}} & 0 & \dots & 0 \\ 0 & 1+K_{2\,\text{tot}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1+K_{n\,\text{tot}} \end{array}\right\| \times$$

$$\times\left\|\begin{array}{cccc} K_1^{(0)} & 0 & \dots & 0 \\ 0 & K_2^{(0)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & K_n^{(0)} \end{array}\right\|. \qquad (2.80)$$

It should be remembered that the coefficients α_{ik} and r_{ik} enter the matrix with their respective signs.

The generalized error matrix is thus a product of two matrices dependent on the coupling coefficients α_{ik} and r_{ik} and a third matrix — the error matrix of the noninteracting subsystems.

Consider the following example. Determine the error matrix of two systems coupled through the plant and the measuring devices and establish the equivalent errors for each interacting subsystem.

We have

$$K_0=\left\|\begin{array}{cc} 1 & \dfrac{K_1\mu_1\alpha_{12}-\mu_1 r_{12}}{1+K_{1\,\text{tot}}} \\ \dfrac{K_2\mu_2\alpha_{21}-\mu_2 r_{21}}{1+K_{2\,\text{tot}}} & 1 \end{array}\right\|^{-1} \left\|\begin{array}{cc} \dfrac{1}{1+K_{1\,\text{tot}}} & 0 \\ 0 & \dfrac{1}{1+K_{2\,\text{tot}}} \end{array}\right\| \times$$

$$\times\left\|\begin{array}{cc} 1 & K_1\alpha_{12} \\ K_2\alpha_{21} & 1 \end{array}\right\| \left\|\begin{array}{cc} 1+K_{1\,\text{tot}} & 0 \\ 0 & 1+K_{2\,\text{tot}} \end{array}\right\| \times \left\|\begin{array}{cc} K_1^{(0)} & 0 \\ 0 & K_2^{(0)} \end{array}\right\|. \qquad (2.81)$$

The inverse preceding the first factor in (2.81) can be found in explicit form. The transpose in our case is

$$K_T=\left\|\begin{array}{cc} 1 & \dfrac{K_2\mu_2\alpha_{21}-\mu_2 r_{21}}{1+K_{2\,\text{tot}}} \\ \dfrac{K_1\mu_1\alpha_{12}-\mu_1 r_{12}}{1+K_{1\,\text{tot}}} & 1 \end{array}\right\|.$$

The determinant of the system is

$$\left|\begin{array}{cc} 1 & \dfrac{K_1\mu_1\alpha_{12}-\mu_1 r_{12}}{1+K_{1\,\text{tot}}} \\ \dfrac{K_2\mu_2\alpha_{21}-\mu_2 r_{21}}{1+K_{2\,\text{tot}}} & 1 \end{array}\right| = 1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})\,(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\text{tot}})\,(1+K_{2\,\text{tot}})}.$$

The inverse may therefore be written as

$$\left\|\begin{array}{cc} \dfrac{1}{1-R_{11}} & \dfrac{K_1\mu_1\alpha_{12}-\mu_1 r_{12}}{(1+K_{1\,\text{tot}}[1-R_{12}])} \\ \dfrac{K_2\mu_2\alpha_{21}-\mu_2 r_{21}}{1+K_{2\,\text{tot}}[1-R_{21}]} & \dfrac{1}{1-R_{22}} \end{array}\right\|. \qquad (2.82)$$

where

$$R_{11}=\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})},$$

$$R_{12}=\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})},$$

$$R_{21}=\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})},$$

$$R_{22}=\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_{21} r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}.$$

This inverse is now multiplied successively by the matrices on its right in equation (2.81). Having performed the multiplication, we find

$$K=\left\| \begin{matrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|, \tag{2.83}$$

where

$$K_{11}=\frac{K_1^{(0)}}{1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}+$$

$$+\frac{K_1^{(0)}(K_1\mu_1\alpha_{12}-\mu_1 r_{12})K_2\alpha_{21}\frac{1+K_{1\,\mathrm{tot}}}{1+K_{2\,\mathrm{tot}}}}{1+K_{1\,\mathrm{tot}}\left[1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{12}=\frac{K_1^{(0)}K_1\alpha_{12}\frac{1+K_{2\,\mathrm{tot}}}{1+K_{1\,\mathrm{tot}}}}{1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}+$$

$$+\frac{K_1^{(0)}[K_1\mu_1\alpha_{12}-\mu_1 r_{12}]}{1+K_{1\,\mathrm{tot}}\left[1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{21}=\frac{K_2^{(0)}[K_2\mu_2\alpha_{21}-\mu_2 r_{21}]}{1+K_{2\,\mathrm{tot}}\left[1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]}+$$

$$+K_2^{(0)}\frac{K_2\alpha_{21}\frac{1+K_{1\,\mathrm{tot}}}{1+K_{2\,\mathrm{tot}}}}{1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}},$$

$$K_{22}=\frac{K_2^{(0)}K_1\frac{a_{12}(1+K_{2\,\mathrm{tot}})}{1+K_{1\,\mathrm{tot}}}(K_2\mu_2 r_{21}-\mu_2 r_{21})}{1+K_{2\,\mathrm{tot}}\left[1-\frac{(K_1\mu_1\alpha_{12}-\mu_1 r_{12})(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]}+$$

$$+\frac{K_2^{(0)}}{1-\frac{(K_1\mu_1\alpha_{12}-\mu_2 r_{12})-(K_2\mu_2\alpha_{21}-\mu_2 r_{21})}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}.$$

Matrix (2.83) is the error matrix of a two-variable system in explicit form.

In particular, if the subsystems are uncoupled, we have $a_{12}=a_{21}=r_{12}=r_{21}=0$ and (2.83) takes the form

$$K_0=\left\| \begin{matrix} K_1^{(0)} & 0 \\ 0 & K_2^{(0)} \end{matrix} \right\|. \tag{2.84}$$

From (2.83) and (2.84) we can estimate the effect of plant- and transducer-coupling on the equivalent errors in each subsystem.

Another interesting case is the error matrix for pure plant coupling or pure transducer coupling (and not mixed coupling, as in the preceding). Putting in (2.83) $r_{12}=r_{21}=0$, we obtain the error matrix for a plant-coupled two-variable system:

$$K=\left\| \begin{matrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|, \tag{2.85}$$

where

$$K_{11}=\frac{K_2^{(0)}}{1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}+ \\ +\frac{K_1^{(0)}K_1K_2\mu_1\alpha_{12}\alpha_{21}\frac{1+K_{1\,\mathrm{tot}}}{1+K_{2\,\mathrm{tot}}}}{1+K_{1\,\mathrm{tot}}\left[1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{12}=\frac{K_1^{(0)}K_1\alpha_{12}\frac{1+K_{2\,\mathrm{tot}}}{1+K_{1\,\mathrm{tot}}}}{1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}+\frac{K_1^{(0)}K_1\mu_1\alpha_{12}}{1+K_{1\,\mathrm{tot}}\left[1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{21}=\frac{K_2^{(0)}K_2\mu_2\alpha_{21}}{1+K_{2\,\mathrm{tot}}\left[1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]}+\frac{K_2^{(0)}K_2\alpha_{21}\frac{1+K_{1\,\mathrm{tot}}}{1+K_{2\,\mathrm{tot}}}}{1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}},$$

$$K_{22}=\frac{K_2^{(0)}\cdot K_1K_2\mu_2\alpha_{12}\alpha_{21}\frac{1+K_{2\,\mathrm{tot}}}{1+K_{1\,\mathrm{tot}}}}{1+K_{2\,\mathrm{tot}}\left[1-\frac{\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]}-\frac{K_2^{(0)}}{1-\frac{K_1K_2\mu_1\mu_2\alpha_{12}\alpha_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}.$$

In the case of transducer coupling, we put $a_{12}=a_{21}=0$ and obtain from (2.83)

$$K=\left\| \begin{matrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{matrix} \right\|, \tag{2.86}$$

where

$$K_{11}=\frac{K_1^{(0)}}{1-\frac{\mu_1\mu_2r_{12}r_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}},$$

$$K_{12}=-\frac{K_1^{(0)}\mu_1r_{12}}{1+K_{1\,\mathrm{tot}}\left[1-\frac{\mu_1\mu_2r_{12}r_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{21}=-\frac{K_2^{(0)}\mu_2r_{21}}{1+K_{2\,\mathrm{tot}}\left[1-\frac{\mu_1\mu_2r_{12}r_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]},$$

$$K_{22}=\frac{K_2^{(0)}}{1-\frac{\mu_1\mu_2r_{12}r_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}}.$$

Examination of expressions (2.83), (2.84), (2.85), and (2.86) suggests a number of general conclusions for multivariable control systems.

The diagonal elements of the matrix correspond to the equivalent errors of the subsystems, while all the other entries represent the effect of the i-th error on the k-th error.

The expressions above indicate that the errors in multivariable systems with coupling are essentially different from the errors in uncoupled systems. For example, take the first-loop error. From (2.83) we have

$$K_{11}=\frac{M_1M_2}{M_1M_2(K_1K_2\mu_1\alpha_{12}-\mu_1r_{12})N_2}K_1^{(0)}+K_1^{(0)}\frac{M_1^2N_1K_2\alpha_{12}}{M_1M_2+K_{1\,\mathrm{tot}}(M_1M_1-N_1N_2)}, \qquad (2.87)$$

where

$$M_1=1+K_{1\,\mathrm{tot}},\quad M_2=1+K_{2\,\mathrm{tot}},$$
$$N_1=K_1\mu_1\alpha_{12}-\mu_1r_{12},$$
$$N_2=K_2\mu_2\alpha_{21}-\mu_2r_{21}.$$

Without coupling $K_{11}=K_1^{(0)}$. If plant coupling is stronger than transducer coupling, i.e., if $\alpha_{ik}>r_{ik}$, the equivalent error K_{11} is greater than $K_1^{(0)}$. In particular, pure plant coupling increases the system error.

Conversely if the coupling coefficient r_{ik} can be so chosen that $r_{ik}>K_i\alpha_{ik}$, an appropriate choice of $K_{2\,\mathrm{tot}}$ will make the error K_{11} less than $K_1^{(0)}$. In particular, *if no plant coupling is imposed*, i.e., $\alpha_{ik}=0$, appropriate choice of the subsystem gains will substantially reduce the errors. This situation obtains in multidimensional servosystems, which are transducer-coupled without plant coupling. The recently developed so-called control-coupled systems are also classified as multidimensional servosystems.

Consider a nondiagonal element of the matrix (2.86):

$$-\frac{K_1^{(0)}\mu_1r_{12}}{1+K_{1\,\mathrm{tot}}\left[1-\frac{\mu_1\mu_2r_{12}r_{21}}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})}\right]}. \qquad (2.88)$$

If the controlled variables are independent, $r_{ik}=0$ and all the elements with r_{ik} vanish. Furthermore, as we have shown in /39/, in single-variable systems increase of each loop gain lowers the system errors and is thus advantageous from this and some other points of view.* It is clear from (2.86) that the nondiagonal elements of the matrix will approach zero as the gain of each control loop is increased indefinitely.

* The effect of gain on the dynamic properties of the system is considered in Chapter Four.

Chapter Three

STRUCTURE OF MULTIVARIABLE CONTROL SYSTEMS

§ 3.1. INTRODUCTORY REMARKS

In the previous chapter we considered multivariable control systems with single-loop subsystems made up of basic unidirectional dynamic elements. The philosophy behind this approach was explained in the preceding. In our analysis of these systems we have established that with regard to the steady-state error they do not differ from ordinary single-loop systems, where an increase in gain improves the accuracy; however, even in this simple case there are substantial differences between multivariable and single-variable control systems. These differences are best illustrated by considering the characteristic equation.

The general transfer function of a closed-loop single-variable system without stabilization which is made up of basic dynamic elements in a single-loop configuration is given by

$$K(p)=\frac{\prod_{l=1}^{n}\frac{K_l}{a_l(p)}}{1+\prod_{l=1}^{n}\frac{K_l}{a_l(p)}}, \tag{3.1'}$$

where $a_i(p)$ is the self-operator of an element. Depending on the exact nature of the elements in the control loop, $a_i(p)$ is a polynomial of first, second, or zeroth degree.

The characteristic equation of the system is written in the form

$$\prod_{l=1}^{n} a_l(p)+K=0, \tag{3.2'}$$

where $K=\prod_{l=1}^{n} K_l$ is the overall system gain. In multivariable control systems, even those with single-loop subsystems, the characteristic equation is a sum of polynomials. It is clear from Chapter Two that the characteristic equation of a multivariable control system can be written as

$$P_0(p)+f_1(K_i)\rho_1(\alpha_{i1})P_1(p)+f_2(K_l)\rho_2(\alpha_{i2})P_2(p)+\ \cdots\ +f_n(K_l)\rho_n(\alpha_{in})P_n(p)=0, \tag{3.3'}$$

where f_i and ρ_i are functions of the loop gain factors and functions of the coupling coefficients between the individual controlled variables. P_i are functions of the self-operators of the individual subsystems.

The effects of gain and coupling on system dynamics should be considered separately, but regardless of the outcome of this analysis it is clear that single-loop configuration does not ensure satisfactory dynamic properties in multivariable systems.

Now, what is the desired structure of multivariable control systems? In other words, what should constitute the foundation for the synthesis of multivariable control systems? In our analysis of single-variable systems /39/, an optimum system was defined as a system which, given the necessary and sufficient number of simple dynamic elements, complied with the specified technical requirements. For a very general class of high-quality control system, the problem of synthesis is reduced to the determination of structures which remain stable at arbitrarily large gain factors and have an infinite closed-loop positive-response bandwidth.

Let us now consider the case of multivariable control. The only general approach to the problem of synthesis of multivariable control system is found in /85/. The author distinguishes between three so-called canonic structures, which differ in the mode of coupling between the individual variables, and the synthesis is based on the following two factors:

(a) $R=r-r_d$, the number of free inputs, and

(b) $D=r_d-n$, the number of inputs which may optimize the process (in respect to a certain criterion) minus the total number of outputs.

It is established /85/ that the above data are insufficient for optimum synthesis and that some additional information is needed. This gap is filled by certain constraints imposed on the system or by the assumption that some of the network elements are known.

Our approach to the problem is essentially different. First, the one-loop configuration is the only permissible, a priori known structure of the starting subsystems; the dynamics of each subsystem is determined by the dynamic properties of the measuring devices, the controlled object (in relation to the particular controlled variable), the corresponding controller, and the amplifiers. This choice of the initial structure is suggested by the very nature of the control process, and these elements always constitute the initial or the starting control loop.

Second, there is a possibility of natural coupling, due to the properties of the controlled object or the load. This may be either direct or cross coupling. Artificial dependence is introduced only if the measuring devices are interconnected in a special way to produce a multidimensional servosystem. As regards other types of artificial coupling between controllers or special load disturbances introduced to ensure, say, noninteraction and certain desirable dynamic properties, they cannot be regarded as known from the start, since they are inherently the outcome of synthesis and not the initial data for synthesis.

The synthesis of multivariable systems, as that of single-variable networks, is based on a number of requirements.

1. Each component system, considered in isolation from the other variables, should allow indefinite increase in gain without losing its stability.

2. The subsystems should theoretically have an infinite closed-loop positive-response bandwidth.

3. Depending on the properties of the controlled object or the problem being considered, we demand that the transient be close to the optimum

for each controlled variable or that it meet a certain optimality criterion for a generalized parameter representing the set of all controlled variables.

Thus, if we know how to build single-variable systems complying with given requirements, the synthesis of multivariable systems reduces to the determination of the dependence of coupling on system structure for the case of subsystems consisting of more than one loop.

Our analysis will proceed as follows. First we shall consider the synthesis of systems where the individual controlled variables are plant-coupled. Then the complexity of the problem will be increased by consideration of load coupling, and finally of combined load and transducer coupling. Multivariable combined-control systems with artificial load coupling introduced to improve the dynamic response of the system are considered separately.

In this chapter we consider systems with plant-coupled controlled variables.

§ 3.2. THE EFFECT OF SUBSYSTEM GAIN ON STABILITY OF MULTIVARIABLE CONTROL SYSTEMS

Let us consider the effect of subsystem gain factors on the stability of a MCS consisting of single-loop subsystems.

In Chapter Two we derived an equation for the j-th controlled variable under these conditions (equation (2.29)). It is written as

$$Y_{j0}=\frac{1}{\Delta_0}\sum_{i=1}^{n}(-1)^{i+j}A_{ij0}(p)\,[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)+g_{ii}(p)f_i(p)]. \qquad (2.29)$$

The characteristic equation of a multivariable control system is

$$\Delta=\begin{vmatrix} a_{11}(p) & a_{12}(p) & \dots & a_{1n}(p)\\ a_{21}(p) & a_{22}(p) & \dots & a_{2n}(p)\\ \dots & \dots & \dots & \dots\\ a_{i1}(p) & a_{i2}(p) & \dots & a_{in}(p)\\ \dots & \dots & \dots & \dots\\ a_{n1}(p) & a_{n2}(p) & \dots & a_{nn}(p)\end{vmatrix}=0, \qquad (3.1)$$

where

$$\left.\begin{aligned} a_{ii}(p)&=D_i(p)R_i(p)Q_i(p)e^{\tau_i p}+K_iK_{i\,\mathrm{a}}\delta_i\mu_i,\\ a_{ik}(p)&=K_iR_i(p)Q_i(p)\alpha_{ik}(p).\end{aligned}\right\} \qquad (3.2)$$

We introduce a new symbol: $\beta_{\tau_{ii}}(p)=D_i(p)R_i(p)Q_i(p)e^{\tau_i p}$, the self-operator of a control loop made of basic elements. In the lagless case, this operator will be denoted $\beta(p)$. We also write

$$R_i(p)Q_i(p)=\gamma_i(p),\quad K_iK_{i\,\mathrm{a}}\delta_i\mu_i=K_{i\,\mathrm{tot}}.$$

In this nomenclature, equation (3.1) takes the form

$$\Delta=\begin{vmatrix} \beta_{\tau 11}(p)+K_{1\,\mathrm{tot}} & K_1\gamma_1(p)\alpha_{12}(p) & \dots & K_1\gamma_1(p)\alpha_{1n}(p)\\ K_2\gamma_2(p)\alpha_{21}(p) & \beta_{\tau 22}(p)+K_{2\,\mathrm{tot}} & \dots & K_2\gamma_2(p)\alpha_{2n}(p)\\ \dots & \dots & \dots & \dots\\ K_n\gamma_n(p)\alpha_{n1}(p) & K_n\gamma_n(p)\alpha_{n2}(p) & \dots & \beta_{\tau nn}(p)+K_{n\,\mathrm{tot}}\end{vmatrix}=0. \qquad (3.3)$$

(a) SYSTEM WITHOUT LAG

Expanding the determinant, we write equation (3.3) in the form

$$F'_{N0}(p)+\sum_{i=1}^{n}K_iF'_{Ni}(p)+\sum_{i,j=1}^{n}K_iK_jF_{Nij}(p)+\ \dots\ +K_1K_2\dots K_n+$$
$$+f[\alpha_{ik}(p)]F_n(p)+f_2[\alpha_{ik}(p)]F_{n-1}(p)+\dots+f_n[\alpha_{ik}(p)]=0, \qquad (3.4)$$

where $F'_{N0}(p)$, F'_{Ni}, F'_{Nij} are polynomials in the variable p, with coefficients independent of the subsystem gains, $F_n(p)$, $F_{n-1}(p)$, ... are polynomials with coefficients independent of K_i, and $\alpha_{ik}(p)$ and $f[\alpha_{ik}(p)]$ are functions of the coupling coefficients.

We now show that under certain conditions increasing the gain of some or all subsystems renders the multivariable control system unstable, and that in multivariable systems with single-loop subsystems there is a contradiction between the feasibility of infinite gain and the stability of the system. We assume the following relationships between the loop gains of the system:

$$\left.\begin{aligned}K_1&=K,\\K_2&=\eta_1K,\\&\dots\dots\\K_n&=\eta_{n-1}K.\end{aligned}\right\} \qquad (3.5)$$

Substituting (3.5) in (3.4), we find

$$F'_{N0}(p)+KF_{N1}(p)+K^2F_{N2}(p)+\ \dots\ +K^n\prod_{i=1}^{n-1}\eta_i+$$
$$+f_1[\alpha_{ik}(p)]F_n(p)+f_2[\alpha_{ik}(p)]F_{n-1}(p)+\dots+f_n[\alpha_{ik}(p)]=0. \qquad (3.6)$$

We divide (3.6) by K^n and write $1/K=m$. Equation (3.6) takes the form

$$m^nF_{N0}(p)+m^{n-1}F_{N1}(p)+m^{n-2}F_{N2}(p)+\ \dots\ +mF_{N,n-1}(p)+\prod_{i=1}^{n-1}\eta_i=0, \qquad (3.7)$$

where

$$F_{N0}(p)=F'_{N0}(p)+f_1[\alpha_{ik}(p)]F_n(p)+\ \dots\ +f_n[\alpha_{ik}(p)].$$

Increasing the gain is equivalent to decreasing m. Our problem thus reduces to investigation of system stability as $m\to0$.

Suppose that in the general case the characteristic equation can be written in the form

$$m^nF_{N0}(p)+m^{n-1}F_{N1}(p)+m^{n-2}F_{N2}(p)+\ \dots\ +mF_{N,n-1}(p)+F_{Nn}(p)=0. \qquad (3.8)$$

Here the subscripts of F denote the degree of the polynomials. We now proceed to determine the conditions under which the roots of equation (3.8) are situated for $m\to0$ in the left-half plane (i.e., to the left of the imaginary axis).

It is clear from our notation that the total number of roots in equation (3.8) is N_0. Let $m\to0$ in equation (3.8). Then N_n out of the total N_0 roots will approach the roots of the equation

$$F_{Nn}(p)=0, \qquad (3.9)$$

which we call the degenerate equation, by analogy with the theory of single-variable control systems /39/. The other $N_0 - N_n$ roots will tend to infinity as $m \to 0$.

Suppose that the degenerate equation $F_{Nn}(p)=0$ satisfies the stability criteria. Then the stability of the entire equation (3.8) will depend on the disposition of the $N_0 - N_n$ roots which recede to infinity as $m \to 0$.

Let us consider the following cases.

Case 1.

$$N_1 = N_0 - 1,\ N_2 = N_0 - 2,\ \ldots,\ N_n = N_0 - n. \tag{3.10}$$

We divide equation (3.8) by m^n and write it in expanded form:

$$\begin{aligned} a_{00}p^{N_0} + a_{01}p^{N_0-1} + a_{02}p^{N_0-2} + \ldots + \frac{1}{m}[a_{10}p^{N_0-1} + a_{11}p^{N_0-2} + \ldots] + \\ + \frac{1}{m^2}[a_{20}p^{N_0-2} + a_{21}p^{N_0-3} + a_{22}p^{N_0-4} + \ldots] + \ldots \\ \ldots + \frac{1}{m^n}[a_{n0}p^{N_0-n} + a_{n1}p^{N_0-n-1} + \ldots + a_{n,\,N_0-n}] = 0. \end{aligned} \tag{3.11}$$

The degenerate equation in this case is

$$a_{n0}p^{N_0-n} + a_{n1}p^{N_0-n-1} + a_{n2}p^{N_0-n-2} + \ldots + a_{n,\,N_0-n} = 0. \tag{3.12}$$

It is implied that the coefficients of the degenerate equation satisfy the stability criteria, since otherwise further analysis is meaningless. Thus for $m \to 0$, $N_0 - n$ roots of equation (3.11) approach the $N_0 - n$ roots of equation (3.12), which by definition lie in the left-half plane.

We now derive an equation which gives the location of the n roots receding to infinity as $m \to 0$. Let

$$p = \frac{q}{m}. \tag{3.13}$$

Substituting (3.13) in (3.11) we find

$$\begin{aligned} a_{00}\frac{q^{N_0}}{m^{N_0}} + a_{01}\frac{q^{N_0-1}}{m^{N_0-1}} + \ldots + a_{10}\frac{q^{N_0-1}}{m^{N_0}} + a_{11}\frac{q^{N_0-2}}{m^{N_0-1}} + \ldots \\ \ldots + a_{20}\frac{q^{N_0-2}}{m^{N_0}} + a_{21}\frac{q^{N_0-3}}{m^{N_0-1}} + \ldots \\ \ldots + a_{n0}\frac{q^{N_0-n}}{m^{N_0}} + a_{n1}\frac{q^{N_0-n-1}}{m^{N_0-1}} + \ldots = 0. \end{aligned} \tag{3.14}$$

Multiplying (3.14) by m^{N_0} and taking $m \to 0$, we find in the limit

$$a_{00}q^{N_0} + a_{10}q^{N_0-1} + a_{20}q^{N_0-2} + \ldots + a_{n0}q^{N_0-n} = 0 \tag{3.15}$$

or, eliminating q^{N_0-n} roots,

$$a_{00}q^n + a_{10}q^{n-1} + a_{20}q^{n-2} + \ldots + a_{n0} = 0. \tag{3.16}$$

We shall refer to equation (3.16) as the auxiliary equation of the first kind. It comprises the leading coefficients of the polynomials in (3.11) and determines the location of the n roots which receded to infinity as $m \to 0$. The roots of this equation move to infinity in the left-half plane if the coefficients of (3.16) comply with the stability criteria.

To sum up, if condition (3.10) is satisfied, the multivariable control system remains stable regardless of an indefinite increase in the subsystem gains, provided that the degenerate equation and the auxiliary equation of the first kind each comply with the stability criteria.

Case 2.

$$N_1 = N_0 - 2, \quad N_2 = N_0 - 4, \ \ldots, \ N_n = N_0 - 2n. \tag{3.17}$$

Equation (3.8) is now written in the form

$$\begin{aligned} a_{00}p^{N_0} + a_{01}^{N_0-1} + a_{02}p^{N_0-2} + \ \ldots \\ \ldots + \frac{1}{m}[a_{10}p^{N_0-2} + a_{11}p^{N_0-3} + a_{13}p^{N_0-4} + \ \ldots] + \\ + \frac{1}{m^2}[a_{20}p^{N_0-4} + a_{21}p^{N_0-5} + \ \ldots] + \\ + \frac{1}{m^n}[a_{n0}p^{N_0-2n} + a_{n1}p^{N_0-2n-1} + \ \ldots \ + a_{n,\,N_0-2n}] = 0. \end{aligned} \tag{3.18}$$

The degenerate equation is

$$a_{n0}p^{N_0-2n} + a_{n1}p^{N_0-2n-1} + \ \ldots \ + a_{n,\,N_0-2n} = 0. \tag{3.19}$$

The degenerate equation is again assumed to satisfy the stability conditions. To establish the stability of the entire system, we have to elucidate the location of the $2n$ roots which recede to infinity as $m \to 0$.

Substituting in (3.18)

$$p = \frac{q}{m^{\frac{1}{2}}}, \tag{3.20}$$

multiplying the equation by $m^{\frac{N_0}{2}}$, and taking the limit as $m \to 0$, we obtain after division by q^{N_0-2n}

$$a_{00}q^{2n} + a_{10}q^{2n-2} + a_{20}q^{2n-4} + \ \ldots \ + a_{n0} = 0. \tag{3.21}$$

Putting $x = q^2$, we rewrite (3.21) in the form

$$a_{00}x^n + a_{10}x^{n-1} + a_{20}x^{n-2} + \ \ldots \ + a_{n0} = 0. \tag{3.22}$$

In our investigation of stability of equation (3.21), we are concerned only with the case when the roots of equation (3.22) are real and negative, since all the other alternatives correspond to unstable systems. Now, if the roots of (3.22) are real and negative, the roots of (3.21) are imaginary. This is a limiting case in the Lyapunov theory, and whether (3.21) is stable or unstable depends on the actual location of the roots of (3.21) when m is small but not zero. Thus, in order to determine the location of the $2n$ roots which recede to infinity as $m \to 0$, only the terms linear in m should be retained in the auxiliary equation, dropping all the higher-order terms.

We now proceed to derive the auxiliary equation for m. Substituting (3.20) in (3.18), we find

$$\begin{aligned} a_{00}\frac{q^{N_0}}{m^{\frac{N_0}{2}}} + a_{01}\frac{q^{N_0-1}}{m^{\frac{N_0-1}{2}}} + a_{02}\frac{q^{N_0-2}}{m^{\frac{N_0-2}{2}}} + \ \ldots \ + a_{10}\frac{q^{N_0-2}}{m^{\frac{N_0-2}{2}+1}} + \\ + a_{11}\frac{q^{N_0-3}}{m^{\frac{N_0-3}{2}+1}} + a_{12}\frac{q^{N_0-4}}{m^{\frac{N_0-4}{2}+1}} + \ \ldots \ + a_{20}\frac{q^{N_0-4}}{m^{\frac{N_0-4}{2}+2}} + a_{21}\frac{q^{N_0-5}}{m^{\frac{N-5}{2}+2}} + \end{aligned}$$

$$+a_{22}\frac{q^{N_0-6}}{m^{\frac{N_0-6}{2}+2}}+\ldots+a_{n0}\frac{q^{N_0-2n}}{m^{\frac{N_0-2n}{2}+2}}+a_{n1}\frac{q^{N_0-2n-1}}{m^{\frac{N_0-2n-1}{2}+2}}+\ldots=0.$$

Multiplying by $m^{\frac{N_0}{2}}$, we obtain

$$\begin{aligned}a_{00}q^{N_0}+m^{\frac{1}{2}}a_{01}q^{N_0-1}+ma_{02}q^{N_0-2}+\ldots+a_{10}q^{N_0-2}+\\+m^{\frac{1}{2}}a_{11}q^{N_0-3}+ma_{12}q^{N_0-4}+\ldots+a_{20}q^{N_0-4}+m^{\frac{1}{2}}a_{21}q^{N_0-5}+\\+ma_{22}q^{N_0-6}+\ldots+a_{n0}q^{N_0-2n}+a_{n1}m^{\frac{1}{2}}q^{N_0-2n-1}+\\+ma_{n2}q^{N_0-2n-2}+\ldots=0.\end{aligned}\tag{3.23}$$

Here $m^{\frac{1}{2}}$ is of the first order of smallness. Dropping the terms of higher order in m, we find

$$\begin{aligned}a_{00}q^{N_0}+m^{\frac{1}{2}}a_{01}q^{N_0-1}+a_{10}q^{N_0-2}+m^{\frac{1}{2}}a_{11}q^{N_0-3}+a_{20}q^{N_0-4}+\\+m^{\frac{1}{2}}a_{21}q^{N_0-5}+\ldots+a_{n0}q^{N_0-2n}+a_{n1}m^{\frac{1}{2}}q^{N_0-2n-1}=0\end{aligned}$$

or dividing by q^{N_0-2n-1}, we finally obtain

$$\begin{aligned}a_{00}q^{2n+1}+m^{\frac{1}{2}}a_{01}q^{2n}+a_{10}q^{2n-1}+m^{\frac{1}{2}}a_{11}q^{2n-2}+a_{20}q^{2n-3}+\\+m^{\frac{1}{2}}a_{21}q^{2n-4}+\ldots+m^{\frac{1}{2}}a_{n1}=0.\end{aligned}\tag{3.24}$$

This is an auxiliary equation of second kind which, in distinction from the auxiliary equation of the first kind discussed in the preceding, is composed of the first two leading terms of the polynomials in equation (3.8), every other coefficient being multiplied by $m^{\frac{1}{2}}$.

The roots which recede to infinity as $m\to 0$ are in the left-half plane if the auxiliary equation of second kind complies with the stability criteria. Let us check that the stability criteria are independent of m. Indeed, the Hurwitz determinant for this case is

$$\begin{vmatrix} m^{\frac{1}{2}}a_{01} & m^{\frac{1}{2}}a_{11} & m^{\frac{1}{2}}a_{21} & \ldots & m^{\frac{1}{2}}a_{n1} & 0 & \ldots & 0\\ a_{00} & a_{10} & a_{20} & \ldots & a_{n0} & 0 & \ldots & 0\\ 0 & m^{\frac{1}{2}}a_{01} & m^{\frac{1}{2}}a_{11} & \ldots & 0 & 0 & \ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \ldots & \ldots & \ldots & 0 & \ldots & m^{\frac{1}{2}}a_{n1}\end{vmatrix}>0.\tag{3.25}$$

We see from (3.25) that $m^{\frac{1}{2}}$ is a common factor for all the elements in every other row and it can be taken outside the determinant. Clearly, $m^{\frac{1}{2}}$ alters the scale of (3.25) but not its sign. In writing the auxiliary equation we may therefore omit the factor $m^{\frac{1}{2}}$ in all the coefficients of this equation. We have thus proved the following proposition.

If condition (3.17) is satisfied (mathematically this means that introduction of the next higher order of m adds 2 to the degree of the equation), the system is stable provided that the degenerate equation obtained from the general equation by putting $m=0$ and the auxiliary equation of second kind comply with the stability criteria.

Case 3. Here introduction of the next higher order of m raises the degree of the equation by 3, i.e.,

$$N_1 = N_0 - 3, \qquad N_2 = N_0 - 6, \ \ldots, \ N_n = N_0 - 3n. \tag{3.26}$$

As in Case 2, we make the substitution

$$p = \frac{q}{m^{\frac{1}{3}}},$$

and write the auxiliary equation in the form

$$a_{00}q^{3n} + a_{10}q^{3n-3} + a_{20}q^{3n-6} + \ \ldots \ + a_{n0} = 0. \tag{3.27}$$

Putting $y = q^3$, we rewrite (3.27) in the form

$$a_{00}y^n + a_{10}y^{n-1} + a_{20}y^{n-2} + \ \ldots \ + a_{n0} = 0. \tag{3.28}$$

Equation (3.28) always has right-half-plane roots, and the system is unstable. Indeed, the only case which requires verification is that of (3.28) with real and negative roots, since otherwise the system is definitely unstable.

Suppose that the coefficients of (3.28) satisfy the conditions of aperiodic stability /39/. Then all its roots are real and negative. To find the roots of (3.27), we make use of the relation

$$q = \sqrt[3]{y}. \tag{3.29}$$

By recalling the properties of binomial equations we conclude that at least one of the three roots of (3.29) is in the right-half plane. Indeed, the roots of an n-th order binomial equation are given by

$$q = \sqrt[n]{|y|}\left(\cos\frac{2\pi k}{n} + j\sin\frac{2\pi k}{n}\right),$$

where $k = 1, 2, \ldots$

In our case $n = 3$ and the three roots are

$$q_{1,2,3} = \sqrt[3]{|y|}\left(\cos\frac{2\pi k}{3} + j\sin\frac{2\pi k}{3}\right) \qquad (k = 1, 2, 3)$$

or

$$q_1 = \sqrt[3]{|y|}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right),$$
$$q_2 = \sqrt[3]{|y|}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right),$$
$$q_3 = \sqrt[3]{|y|}\cdot 1.$$

One of these roots, $\sqrt[n]{|y|}$, is positive.

Since equation (3.28) has n roots, at least n of the $3n$ roots of the auxiliary equation (3.27) are in the right-half plane and the system is unstable. To sum up, if we can find two adjoining polynomials with degrees differing by more than two (three or more) and the higher-order polynomial is multiplied by m to the higher power, the system is unstable.

Case 4. In this case the difference in the degrees of the polynomials is variable.

We have already established that if the difference in the degrees of any two adjoining polynomials is three or more, the system is unstable. We should therefore concentrate only on the case when the difference in the degrees of adjoining polynomials is either one or two. The corresponding equation can be written in the following form:

$$m^n F_{N_0}(p) + m^{n-1}F_{N-1}(p) + m^{n-2}F_{N_0-3} + m^{n-3}F_{N_0-5}(p) + \\ + m^{n-4}F_{N_0-7}(p) + \ldots + F_{N_0-n}(p) = 0. \qquad (3.30)$$

We shall show that the polynomials must be arranged in the order of increasing difference in degrees, since otherwise the system is unstable. It of course suffices to show that violation of this rule in any particular case results in system instability. Consider the simple equation

$$m^2 F_{N_0}(p) + mF_{N_0-2}(p) + F_{N_0-3}(p) = 0. \qquad (3.31)$$

Here the polynomial of degree N_0 is followed by a polynomial of degree N_0-2 and then by a polynomial of degree N_0-3, i.e., in this three-membered equation the degree of the polynomials decreases first by 2 and then by 1. We write (3.31) in expanded form:

$$a_{00}p^{N_0} + a_{01}p^{N_0-1} + a_{02}p^{N_0-2} + \ldots + \frac{1}{m}[a_{10}p^{N_0-2} + a_{11}p^{N_0-3} + \ldots] + \\ + \frac{1}{m^2}[a_{20}p^{N_0-3} + a_{21}p^{N_0-4} + \ldots] = 0. \qquad (3.32)$$

Substituting in (3.32)

$$p = -\frac{q}{m^{\frac{1}{2}}}, \qquad (3.33)$$

we find

$$a_{00}\frac{q^{N_0}}{m^{\frac{N_0}{2}}} + a_{01}\frac{q^{N_0-1}}{m^{\frac{N_0-1}{2}}} + a_{02}\frac{q^{N_0-2}}{m^{\frac{N_0-2}{2}}} + \ldots + a_{10}\frac{q^{N_0-2}}{m^{\frac{N_0}{2}}} + a_{11}\frac{q^{N_0-3}}{m^{\frac{N_0-1}{2}}} + \ldots \\ \ldots + a_{20}\frac{q^{N_0-3}}{m^{\frac{N_0+1}{2}}} + a_{21}\frac{q^{N_0-4}}{m^{\frac{N_0}{2}}} + a_{22}\frac{q^{N_0-5}}{m^{\frac{N_0-1}{2}}} + \ldots = 0. \qquad (3.34)$$

We multiply (3.34) by $m^{\frac{N_0+1}{2}}$ in order to eliminate the m in the denominator, and write

$$m^{\frac{1}{2}}a_{00}q^{N_0} + ma_{01}q^{N_0-1} + m^{\frac{3}{2}}a_{02}q^{N_0-2} + \ldots + m^{\frac{1}{2}}a_{10}q^{N_0-2} + \\ + ma_{11}q^{N_0-3} + m^{\frac{3}{2}}a_{12}q^{N_0-4} + \ldots + a_{20}q^{N_0-3} + m^{\frac{1}{2}}a_{21}q^{N_0-4} + \\ + ma_{22}q^{N_0-5} + \ldots = 0. \qquad (3.35)$$

Equation (3.35) has coefficients of various orders of smallness. Suppose that we decide to retain terms with $m^{\frac{1}{2}}$; dropping terms of higher order of smallness, we find

$$m^{\frac{1}{2}}a_{00}q^{N_0}+m^{\frac{1}{2}}a_{02}q^{N_0-2}+a_{20}q^{N_0-3}+m^{\frac{1}{2}}a_{21}q^{N_0-4}=0. \tag{3.36}$$

The coefficients of equation (3.36) do not comply with the stability criteria for two reasons. First, the coefficient of q^{N_0-1} is zero and, second, equation (3.36) may be written in the form

$$m^{\frac{1}{2}}[a_{00}q^{N_0}+a_{02}q^{N_0-2}+a_{21}q^{N_0-4}]+a_{20}q^{N_0-3}=0 \tag{3.37}$$

or

$$N_0-N_0+3=3, \tag{3.38}$$

and according to the preceding rule, it has at least one right-half-plane root for small m.

If terms to the order of m are retained in (3.35), condition (3.38) remains in force and the system is unstable as before. We have thus proved a highly important condition: the polynomials should be arranged in such a sequence that the difference in their degrees increases.

Let us consider the derivation of the auxiliary equation when the above condition is satisfied. It is clear that a difference of one in the degrees of adjoining polynomials is permissible only between the first and the second polynomials, and further down the series the difference must be two. This follows directly from the rule that we have just proved, which can be called the property of declining degrees.

We start with the equation

$$m^nF_{N_0}(p)+m^{n-1}F_{N_0-1}(p)+m^{n-2}F_{N_0-3}(p)+\ldots$$
$$\ldots+mF_{N_0-\frac{n}{2}}(p)+F_{N_0-\frac{n}{2}+1}(p)=0. \tag{3.39}$$

Writing (3.39) in expanded form and substituting the variables according to (3.33), we find after simple manipulations

$$a_{00}\frac{q^{N_0}}{m^{\frac{N_0}{2}}}+a_{01}\frac{q^{N_0-1}}{m^{\frac{N_0-1}{2}}}+a_{02}\frac{q^{N_0-2}}{m^{\frac{N_0-2}{2}}}+\ldots$$
$$\ldots+a_{10}\frac{q^{N_0-1}}{m^{\frac{N_0+1}{2}}}+a_{11}\frac{q^{N_0-2}}{m^{\frac{N_0}{2}}}+\ldots+a_{20}\frac{q^{N_0-3}}{m^{\frac{N_0+1}{2}}}+a_{21}\frac{q^{N_0-4}}{m^{\frac{N_0}{2}}}+$$
$$+a_{22}\frac{q^{N_0-5}}{m^{\frac{N_0-2}{2}}}+\ldots+a_{30}\frac{q^{N_0-5}}{m^{\frac{N_0+1}{2}}}+a_{31}\frac{q^{N_0-6}}{m^{\frac{N_0}{2}}}+a_{32}\frac{q^{N_0-7}}{m^{\frac{N_0-1}{2}}}+\ldots$$
$$\ldots+a_{N_0-n-1,0}\frac{q^{N_0-\frac{n}{2}-1}}{m^{\frac{N_0+1}{2}}}+a_{N_0-n-1,1}\frac{q^{N_0-\frac{n}{2}-2}}{m^{\frac{N_0}{2}}}+$$
$$+a_{N_0-n-1,2}\frac{q^{N_0-\frac{n}{2}-3}}{m^{\frac{N_0-1}{2}}}+\ldots=0. \tag{3.40}$$

Multiplying (3.40) by $m^{\frac{N_0+1}{2}}$ and retaining terms of the order $m^{\frac{1}{2}}$, we obtain the auxiliary equation

$$m^{\frac{1}{2}}a_{00}q^{N_0}+a_{10}q^{N_0-1}+m^{\frac{1}{2}}a_{11}q^{N_0-2}+a_{20}q^{N_0-3}+m^{\frac{1}{2}}a_{21}q^{N_0-4}+$$
$$+a_{30}q^{N_0-5}+\ldots+a_{N_0-n-1,0}q^{N_0-n-1}+m^{\frac{1}{2}}a_{N_0-n-1,1}q^{N_0-n-2}=0. \qquad (3.41)$$

The small quantity $m^{\frac{1}{2}}$ clearly does not influence the stability conditions, since it multiplies all the odd terms of the equations. For this reason, $m^{\frac{1}{2}}$ can be omitted from the coefficients in writing the auxiliary equation. It is clear from (3.41) that the auxiliary equation comprises the coefficient of the first term of the leading polynomial and the coefficients of the first two leading terms of all the subsequent polynomials.

Dividing (3.41) by q^{N_0-n-2} and dropping the factor $m^{\frac{1}{2}}$, we find

$$a_{00}q^{n+2}+a_{10}q^{n-1}+a_{11}q^n+a_{20}q^{n-1}+a_{21}q^{n-2}+\ldots$$
$$\ldots+a_{N_0-\frac{n}{2}-1}q+a_{N_0-\frac{n}{2}-2}=0. \qquad (3.42)$$

This is an auxiliary equation of third kind. As an example, we write the auxiliary equation of the third kind for $n=2$. Thus

$$a_{00}q^4+a_{10}q^3+a_{11}q^2+a_{21}q+a_{21}=0. \qquad (3.43)$$

We have thus established under what conditions the subsystem gains can be increased and what conditions are to be satisfied by the coefficients of the general characteristic equation in order for the system not to lose its stability.

We now return to equation (3.6), to determine the structure of the subsystems and to summarize our analysis.

In (3.6)

$$F_{N_0}=\prod_{i=1}^{n}D_i(p)\,Q_i(p)\,R_i(p). \qquad (3.44)$$

This is a product of the products of the self-operators of the elements in a single-loop subsystem:

$$F_{N1}=\sum_{i=1}^{n}K_{j\,\mathrm{tot}}\prod_{j=1}^{n}D_j(p)\,Q_j(p)\,R_j(p)+M(p), \qquad (3.45)$$

where $M(p)$ is a polynomial of degree which is definitely less than the degree of the first term in (3.45) by an amount equal to the degree of $D_i(p)$. Similarly,

$$F_{N2}(p)=\sum K_iK_j\prod_{\substack{\eta=1\\ \eta\neq i,\,j}}^{n}D_\eta(p)\,Q_\eta(p)\,R_\eta(p)+M_1(p)\,, \qquad (3.46)$$

i.e., each successive polynomial contains one product $D(p)Q(p)R(p)$ less than its predecessor. Hence it follows that in our case $D(p)Q(p)R(p)$ is at most of second degree.

Our analysis of the simple basic structure leads to the following conclusions.

1. If the self-operator of each loop with basic elements is of degree 1, all the gains can be increased simultaneously without loss of stability. The degenerate equation and the auxiliary equation of the first kind should each satisfy the stability conditions.

2. If the self-operator of each loop with basic elements is of degree 2, all the gains can be increased simultaneously without loss of stability. The degenerate equation and the auxiliary equation of the second kind should each satisfy the stability conditions.

3. If the self-operator of each loop with basic elements is of degree 3 or higher, an increase of one, several, or all loop gains invariably leads to loss of stability. There is consequently a contradiction between the feasibility of gain increase and the stability of the system, similar to that observed in single-loop systems with a self-operator of degree higher than two.

4. If the self-operators of the different loops in a multivariable system are of different degrees, the gain of none of the loops with self-operators of degree higher than 2 can be increased without losing the stability of the system as a whole.

5. The structure of the multivariable control system should satisfy the rule of declining degrees. A system with first v terms showing a difference of 1 in their degrees and the next $n-v$ terms a difference of 2 obviously meets this criterion.

(b) SYSTEM WITH LAG

We now return to the starting set of equations. We shall try to establish the configuration of a multivariable control system whose subsystems are made up of basic elements plus lags. The set of equations in this case is written in the following form:

$$(\beta_{ii\tau}(p)+K_{i\,\mathrm{tot}})Y_i+K_i\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_i(p)\alpha_{ik}(p)Y_k(p)= =K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)\qquad(i=1,\,2,\,\ldots,\,n). \tag{3.47}$$

The characteristic equation can be written in the form

$$\Delta=\begin{vmatrix} \beta_{11}(p)\,e^{\tau_1 p}+K_{1\,\mathrm{tot}} & K_1\gamma_1(p)\alpha_{12}(p) & \ldots & K_1\gamma_1(p)\alpha_{1n}(p)\\ K_2\gamma_2(p)\alpha_{21}(p) & \beta_{22}(p)\,e^{\tau_2 p}+K_{2\,\mathrm{tot}} & \ldots & K_2\gamma_2(p)\alpha_{2n}(p)\\ \cdots\cdots\cdots & \cdots\cdots\cdots & \ldots & \cdots\cdots\cdots\\ K_n\gamma_n(p)\alpha_{n1}(p) & K_n\gamma_n(p)\alpha_{n2}(p) & \ldots & \beta_{nn}(p)\,e^{\tau_n p}+K_{n\,\mathrm{tot}} \end{vmatrix}=0. \tag{3.48}$$

We have already seen (Chapter Two) that the steady-state error in the i-th controlled variable decreases with the increase in the i-th loop gain in lagged systems, too. It is easily shown that the results of the previous subsection can be completely extended to multivariable control systems where some or all subsystems contain lags. We omit the proof*, since it in fact amounts to repetition of the previous manipulations. We shall only concentrate on the new properties which are attributed to the introduction of lags.

* The validity of this proposition follows from the synthesis of lagged systems remaining stable at infinite gain, which is described in the sequel.

We now prove the following proposition: if there exists at least one pair of coupling coefficients $\alpha_{ik}(p)$ and $\alpha_{ki}(p)$ such that $\alpha_{ik}(p)\alpha_{ki}(p)$ is of higher order in p than $D_i(p)D_k(p)$ is, and time lag is provided in the i-th or the k-th loop, the system is structurally unstable.

Indeed, for the system to become structurally unstable in this case, it suffices to omit the leading term from the characteristic equation.

As we easily see from (3.48) and the procedure for the construction of the characteristic equation for the entire multivariable control system, the maximum value of τ is equal to the sum of all the lags $\tau = \sum_{i=1}^{n} \tau_i$ and a term $e^{p\sum_{i=1}^{n}\tau_i}$ will precede the product $\prod_{i=1}^{n} \beta_{ii}(p)$. Since the quasipolynomials entering the characteristic equation include the quasipolynomial

$$\prod_{i=1,2,\ldots,i-1} \beta_{ii\tau}(p)\alpha_{ik}(p)\alpha_{ki}(p) \prod_{k=i+1\ldots n} \beta_{ii\tau}(p),$$

the characteristic equation will lose its leading term if $\alpha_{ik}(p)\alpha_{ki}(p)$ is of higher degree than $\beta_{ii}(p)\beta_{kk}(p)$ is, and the system will become structurally unstable.

§ 3.3. STRUCTURE OF LAGLESS MULTIVARIABLE CONTROL SYSTEMS WITH INFINITE-GAIN STABILITY

Under real conditions, the self-operators of the subsystems may be of higher than second degree. It follows from § 3.1 that in this case an increase in one or several gain parameters of loops with self-operators of degree higher than 2 will inevitably lead to loss of stability. We thus have the following problem: synthesize a multivariable control system which would be inherently free from the contradiction between stability and precision.

Let the self-operator of the subsystem for one of the controlled variables, say Y_1, be of degree $V_1 > 2$. An increase in gain of this loop inevitably leads to instability of the entire system. This conclusion follows from the preceding results, but it can also be verified directly. Additional proof of this fact will be quite useful in the sequel, and we therefore reproduce it here in detail.

Developing the determinant (3.3) with respect to the first column, we find

$$[\beta_{11}(p) + K_{1\,\text{tot}}]\Delta_{11} + K_2R_2(p)Q_{21}(p)\alpha_{21}(p)\Delta_{21} + \cdots \\ \cdots + K_nR_n(p)Q_n(p)\alpha_{n1}(p)\Delta_{n1} = 0, \qquad (3.49)$$

where Δ_{i1} are the cofactors of the corresponding elements in the first column. The first term in (3.49) contains p to the highest order, since it carries the fewest mutual-coupling coefficients, which are either constants or operators of degree less than the degree of the self-operators of the individual loops. Equation (3.49) can thus be written in the form

$$F_{N2}(p) + K_{1\,\text{tot}}F_{N1}(p) = 0, \qquad (3.50)$$

where the subscripts F designate the degree of the polynomials $F_{N2}(p)$ and $F_{N1}(p)$. Obviously, $N_2 = N_1 + V_1$. Hence it follows that if $V_1 > 2$, the increase in $K_{1\,\text{tot}}$ will immediately result in system instability. It is also obvious that the condition $N_2 - N_1 \leqslant 2$ is satisfied if the $(N_1 - 2)$-th derivative is introduced into the first loop. Indeed, if the $(V_1 - 2)$-th derivative is introduced into the first loop, equation (3.50) takes the form

$$F_{N2}(p) + K_{1\,\text{tot}}(p^{V_1-2}+1)F_{N1}(p) = 0. \tag{3.51}$$

Here $N_2 - N_1 \leqslant 2$. Generalizing this result to the case when the self-operators of each control loop are of degree V_i, we come to the conclusion that stability can be ensured for any $K_{i\,\text{tot}}$ if the $(V_i - 2)$-th derivative is introduced into each loop with $V_i > 2$. This ensures the condition $N_2 - N_1 \leqslant 2$ in the i-th loop.

Now, is it necessary to introduce, besides the $(V_i - 2)$-the derivative, all the lower-order derivatives as well, down to the first derivative?

Before answering this question, let us derive an expression for the degenerate equation, assuming that the $(V_i - 2)$-th derivative has been introduced into each loop with $V_i > 2$. It is easily seen that the degenerate equation has the form

$$(p^{V_1-2}+1)(p^{V_2-2}+1)\ldots(p^{V_i-2}+1)F_{Nn}(p) = 0. \tag{3.52}$$

Regardless of the form of the polynomial $F_{Nn}(p)$, the system is obviously unstable if any $V_k > 3$. Hence it follows that stability of the degenerate equation can be ensured if for $V_k > 3$ all the lower-order derivatives, down to the first derivative, are introduced together with the $(V_i - 2)$-th derivative.

We thus come to the following conclusion. A system with n plant-coupled controlled variables can be stabilized with respect to each controlled variable for any gain value. To this end all the derivatives from $(V_i - 2)$-th down to the first inclusive should be introduced into the corresponding loop (V_i is the order of the self-operator of the i-th loop).

§ 3.4. ALTERNATIVE SOLUTION

In the preceding section we dealt with the synthesis of structures that retained their stability at infinite gain. This necessitated the introduction of ideal derivatives of various orders into the system. We shall see from what follows (and incidentally this is also known from the literature /39/) that in principle real derivatives of any order can be made arbitrarily close to the ideal. This approach, however, can be recommended in practice only if no other more convenient alternative is open to us. In this section we describe a synthesis procedure which achieves the same effect (i.e., indefinite increase of gain without loss of stability) but does not resort to ideal derivatives.

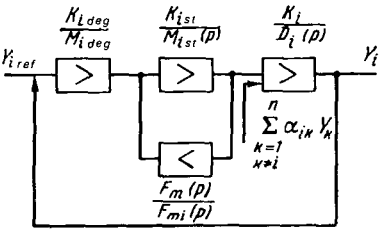

FIGURE 3.1. The i-th subsystem with a stabilizer.

It is clear from the outset that the single-loop configuration is no longer adequate for the subsystems. Figure 3.1 is a block diagram of the i-th subsystem in a multivariable control system. We introduce the following nomenclature for the i-th subsystem:

$M_i(p)D_i(p)$ = the self-operator of the subsystem, ignoring the stabilizer;

$\frac{F_{ni}(p)}{F_{mi}(p)}$ = the operator of the additional element introduced as internal feedback in the subsystem (we call this additional path the stabilizer);

$F_{ni}(p)$ and $F_{mi}(p)$ = polynomials in the operator p;

$K_{i\,st}$ = the gain of the stabilized section, i.e., the part of the forward path embraced by the stabilizer;

$K_{i\,deg}$ = the gain of the unstabilized section outside the stabilizer loop;

K_i' = the plant gain for the i-th controlled variable;

$M_{i\,st}(p)$ = the self-operator of the stabilized part of the controller;

$M_{i\,deg}(p)$ = the self-operator of the unstabilized section.

Clearly $K_{i\,tot}=K_{i\,st}K_i'K_{i\,deg}$.

Now, suppose that the plant has n controlled variables and there are correspondingly n control networks. As before, we assume that the controlled variables are interconnected through the plant, the coupling coefficients being $\alpha_{ik}(p)$. The constraints on $\alpha_{ik}(p)$ are the same as in the preceding. Automatic control can be described by the following set of differential equations:

$$\begin{aligned}&\{D_i(p)M_{i\,deg}(p)[M_{i\,st}(p)F_{mi}(p)+K_{i\,st}F_{ni}(p)]+K_{i\,tot}F_{mi}(p)\}Y_i(p)=\\&=K_{i\,tot}F_{mi}(p)Y_{i\,ref}(p)-K_i'[M_{i\,st}(p)F_{mi}(p)+K_{i\,st}F_{ni}(p)]\times\\&\times M_{i\,deg}(p)\left[\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)+f_k(p)\right]\quad(i=1,\ 2,\ \ldots,\ n).\end{aligned}\tag{3.53}$$

For the sake of convenience we put

$$\left.\begin{aligned}\Pi_i(p)&=D_i(p)M_{i\,deg}(p)M_{i\,st}(p)F_{mi}(p),\\B_i(p)&=D_i(p)M_{i\,deg}(p)F_{ni}(p)+K_{i\,deg}F_{mi}(p),\\D_i(p)&=K_i'M_{i\,st}(p)F_{mi}(p).\end{aligned}\right\}\tag{3.54}$$

The degree of the operator $\Pi_i(p)$ is the degree of the i-th self-operator plus the degree of the denominator of the stabilizer operator. The degree of the operator $B_i(p)$ is the degree of the self-operator of the unstabilized controller plus the degree of the plant operator and the degree of the numerator of the stabilizer operator. The degree of $D_i(p)$ is the degree of the self-operator of the stabilized controller plus the degree of the denominator of the stabilizer operator.

In our new nomenclature, the equations can be written in the form

$$\begin{aligned}&\{\Pi_i(p)+K_{i\,st}B_i(p)]Y_i(p)+\\&+[D_i(p)+K_{i\,st}K_i'F_{ni}(p)]M_{i\,deg}(p)\left[\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)-f_k\right]=\\&=K_{i\,tot}F_{mi}(p)Y_{i\,ref}(p)\quad(i=1,\ 2,\ \ldots,\ n).\end{aligned}\tag{3.55}$$

The characteristic determinant of (3.55) is

$$\begin{vmatrix} A_{11} & A_{12} \dots A_{1n} \\ A_{21} & A_{22} \dots A_{2n} \\ \dots & \dots \\ A_{n1} & A_{n2} \dots A_{nn} \end{vmatrix}, \tag{3.56}$$

where

$$A_{ii} = \Pi_i(p) + K_{i\,st}\ B_i(p)$$

and

$$A_{ik} = [D_i(p) + K_{i\,st}\ K_i' F_{ni}(p)]\, M_{i\,deg}(p)\, \alpha_{ik}(p).$$

Expanding (3.56), we obtain the characteristic equation

$$F_{N0}(p) + K_{st}\ F_{N1}(p) + K_{st}^2\ F_{N2}(p) + \dots + K_{st}^n\ F_{Nn}(p) = 0, \tag{3.57}$$

where $F_{N0}(p)$ includes the product of all $\Pi_i(p)$ and all other products in which K_{st} *does not enter as a factor.* The polynomial $F_{N0}(p)$ clearly contains p to the highest order, and it determines the degree of the characteristic equation.

The polynomial $F_{Ni}(p)$ is a sum of the products of $B_i(p)$ and $\prod_{\substack{k=1 \\ k \neq i}}^{n} \Pi_n(p)$; $F_{Ni}(p)$ also includes all other terms which depend on the coupling coefficients $\alpha_{ik}(p)$ and appear as a factor before K_{st} to the power of 1.

All the successive terms in (3.57) are formed according to the same rule; the higher the subscript N, the fewer $\Pi_i(p)$ appear in the product. The last term in (3.57), having K^n as its coefficient, consists of the product $\prod_{i=1}^{n} B_i(p)$ plus terms dependent on $\alpha_{ik}(p)$ which appear as a factor before K_{st}^n.

Suppose that each control loop with its stabilizer form an isolated network which retains its stability as the gain is increased indefinitely. Then, as it follows from the construction of the polynomials in (3.57), the difference in the degrees of two adjoining polynomials cannot be greater than 2.

We thus arrive at the following procedure for the synthesis of multivariable control systems with infinite-gain stability: the gain of each subsystem in a system with n mutually coupled (through the plant) controlled variables can be increased indefinitely without causing instability of any of the subsystems or the system as a whole if and only if

(a) each subsystem, treated in isolation from other controlled variables, remains stable at arbitrarily high gains, and

(b) the degenerate equation and the auxiliary equations of the first, second, and third kind of the entire multivariable system each comply with the stability criteria.

§ 3.5. LAGGED MULTIVARIABLE SYSTEMS WITH INFINITE-GAIN STABILITY

Let us now try to extend the results of previous sections concerning the synthesis of multivariable control systems with infinite-gain stability to multivariable systems with time lags.

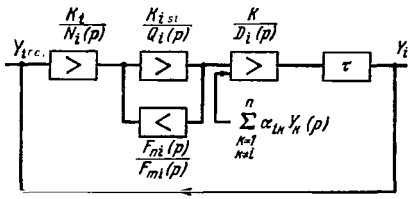

FIGURE 3.2. The i-th subsystem with a stabilizer and a lag element.

Figure 3.2 is a block diagram of a lagged multivariable system with time lag. Part of the system is stabilized by a feedback element with a transfer function $\frac{F_{ni}(p)}{F_{mi}(p)}$. It is assumed that the stabilized section is lagless. We will now establish the properties of the stabilizer and the stabilized section which permit indefinitely increasing the local gain and hence of the total system gain.

A structure shown in Figure 3.2 is described by the following set of differential equations:

$$\begin{aligned}&[P_i(p)e^{\tau_i p}+K_{i\,st}S_i(p)e^{\tau_i p}+K_{i\,tot}F_{mi}(p)]Y_i(p)+\\&+K_i[R_i(p)+K_{i\,st}N_i(p)F_{ni}(p)]\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=\\&=K_i[R_i(p)+K_{i\,st}N_i(p)F_{ni}(p)][Y_{i\,ref}(p)+f_i(p)]\\&(i=1,\ 2,\ \ldots,\ n),\end{aligned}\tag{3.58}$$

where

$$\left.\begin{aligned}P_i(p)&=D_i(p)N_i(p)F_{mi}(p)Q_i(p),\\S_i(p)&=D_i(p)N_i(p)F_{ni}(p),\\R_i(p)&=N_i(p)F_{mi}(p)Q_i(p),\end{aligned}\right\}\tag{3.59}$$

$N_i(p)$ = the self-operator of the unstabilized section of the i-th subsystem;
$Q_i(p)$ = the self-operator of the stabilized section of the i-th subsystem;
K_i = the gain of the unstabilized section of the i-th subsystem;
$K_{i\,st}$ = the gain of the elements in the stabilizer loop;
$K_{i\,tot}=K_{i\,st}K_{i\,deg}$.
The characteristic equation is

$$\begin{vmatrix}A_{11}&A_{12}&\ldots&A_{1n}\\A_{21}&A_{22}&\ldots&A_{2n}\\\cdots&\cdots&\cdots&\cdots\\A_{n1}&A_{n2}&\ldots&A_{nn}\end{vmatrix}=0.\tag{3.60}$$

The expressions for A_{ij} ($i=1,\ 2,\ldots;\ j=1,\ 2,\ldots$) have the form

$$\begin{aligned}A_{ii}&=P_i(p)e^{\tau_i p}+K_{i\,st}S_i(p)e^{\tau_i p}+K_{i\,tot}F_{im}(p),\\A_{ij}&=K_i[R_i(p)+K_{i\,st}N_i(p)F_{ni}(p)]\alpha_{ij}(p).\end{aligned}$$

In the following we assume that the gains of the stabilized elements in the various subsystems are either equal to one another or are related by

$$K_{i\,st}=\eta_i K_{1\,st}=\eta_i K_{st}.$$

The characteristic equation is thus written in the form

$$\prod_{i=1}^{n} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + K_{st}\left\{\prod_{i=1}^{n} S_i(p) C_1^{n-1} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{00}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{0n}(p)\right\} + K_{st}^2\left\{\sum_{i,k=1}^{n} S_1(p) S_k(p) C_n^{n-2} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{10}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{1n}(p)\right\} + \ldots + K_{st}^n\left\{\prod_{i=1}^{n} S_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{n0}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{nn}(p)\right\} + F(p) = 0, \tag{3.61}$$

where $F(p)$ is a polynomial independent of K_{st} and τ. The polynomials $D_{ik}(p)$ are obviously of lower order in p than the polynomials in the first term in braces. Dividing (3.61) through by K_{st}^n we put

$$\frac{1}{K_{st}^i} = m^i.$$

Equation (3.61) is thus written in the form

$$m^n\left\{\prod_{i=1}^{n} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + F(p)\right\} + m^{n-1}\left\{\sum_{i=1}^{n} S_i(p) C_n^{n-1} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{00}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{0n}(p)\right\} + m^{n-2}\left\{\sum_{i,k=1}^{n} S_i(p) S_k(p) C_n^{n-2} P_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{10}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{1n}(p)\right\} + \ldots + \prod_{i=1}^{n} S_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{n0}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{nn}(p) = 0. \tag{3.62}$$

The degenerate equation is obtained by taking $m = 0$ in (3.62):

$$\prod_{i=1}^{n} S_i(p)\, e^{\sum_{i=1}^{n}\tau_i p} + D_{n0}(p)\, e^{\sum_{i=2}^{n}\tau_i p} + \ldots + D_{nn}(p) = 0. \tag{3.63}$$

Suppose that the degenerate equation (3.63) can be made to satisfy the stability conditions by appropriate choice of the stabilizer parameters $F_{ni}(p)$ and $F_{mi}(p)$ (otherwise, further analysis is meaningless). The stability of the entire system is dependent on the location of the roots which recede to infinity as $m \to 0$.

First we should establish the presence of the leading term in equation (3.62). According to the theorem of structural stability proved in the previous section, we know that equation (3.62) has a leading term if for each pair

of coupling coefficients the polynomial $\alpha_{ik}(p)\alpha_{ki}(p)$ is of lower degree than the polynomial $D_i(p)D_k(p)$ is. In what follows we assume that this condition is satisfied.

Let us find the number and the nature of the roots which go to infinity as $m \to 0$. Dividing equation (3.62) by $m^n e^{\sum_{i=1}^{n} \tau_i p}$, we find

$$\prod_{i=1}^{n} P_i(p) + F(p)\, e^{-\sum_{i=1}^{n} \tau_i p} + \frac{1}{m}\left[\sum_{i=1}^{n} S_i(p)\, C_n^{n-1} P_i(p) + \right.$$
$$\left. + D_{00}(p)\, e^{-\tau_i p} + \ldots + D_{0n}(p)\, e^{-\sum_{i=1}^{n} \tau_i p}\right] +$$
$$+ \frac{1}{m^2}\left[\sum_{i,\,k=1}^{n} S_i(p)\, S_k(p)\, C_n^{n-2} P_i(p) + \right.$$
$$\left. + D_{10}(p)\, e^{-\tau_i p} + \ldots + D_{1n}(p) e^{-\sum_{i=1}^{n} \tau_i p}\right] + \ldots$$
$$\ldots + \frac{1}{m^n}\left[\prod_{i=1}^{n} S_i(p) + D_{n0}(p)\, e^{\tau_i p} + \ldots + D_{nn}(p)\, e^{-\sum_{i=1}^{n} \tau_i p}\right] = 0. \qquad (3.64)$$

Equation (3.64) can be expanded in the form

$$A_{00}p^{N_0} + A_{01}p^{N_0-1} + \ldots + (B_{00}p^{\beta_{00}} + B_{01}p^{\beta_{00}-1} + \ldots)\, e^{-\sum_{i=1}^{n} \tau_i p} +$$
$$+ \frac{1}{m}\left[A_{10}p^{N_1} + A_{11}p^{N_1-1} + \ldots + (B_{10}p^{\beta_{10}} + B_{11}p^{\beta_{10}-1} + \ldots)e^{-\tau_i p} + \ldots\right] +$$
$$+ \frac{1}{m^2}\left[A_{20}p^{N_2} + A_{21}p^{N_2-1} + \ldots + (B_{20}p^{\beta_{20}} + B_{21}p^{\beta_{20}-1} + \ldots)e^{-\tau_i p} + \ldots\right] +$$
$$+ \ldots + \frac{1}{m^n}\left[A_{n0}p^{N_n} + A_{n1}p^{N_n-1} + \ldots\right.$$
$$\left. \ldots + (B_{n0}p^{\beta_{n0}} + B_{n1}p^{\beta_{n0}-1} + \ldots)\, e^{-\tau_i p} + \ldots\right] = 0, \qquad (3.65)$$

where $N_0, N_1, \ldots$ are the degrees of the polynomials associated with the corresponding powers of m in (3.65). Let

$$N_0 - N_1 = 1, \quad N_0 - N_2 = 2, \ \ldots, \ N_0 - N_n = n. \qquad (3.66)$$

In other words, the degree of each successive sum of polynomials is one less than that of the preceding sum.

We make the substitution

$$p = \frac{q}{m}. \qquad (3.67)$$

Inserting (3.66) in (3.65), multiplying by m^N and putting $m = 0$, we obtain after some manipulations

$$A_{00}q^n + A_{10}q^{n-1} + \ldots + A_{n0} = 0. \qquad (3.68)$$

There are n roots which go to infinity for $m \to 0$, and their location on the root plane is determined by the coefficients in (3.68).

All the roots of equation (3.68) recede to infinity in the left-half plane if and only if the coefficients of this equation satisfy the Routh—Hurwitz criteria.

We thus come to the conclusion that a system with constraint (3.66) is stable if the degenerate transcendental equation and equation (3.68), which by analogy with the preceding is called an auxiliary equation of the first kind, both satisfy the stability conditions. Our problem is thus to choose the stabilizer transfer function and its location in the system ensuring

$$N_0 - N_i = i.$$

Let us now consider the case

$$N_0 - N_1 = 2, \quad N_0 - N_2 = 4, \quad \ldots, \quad N_0 - N_n = 2n. \tag{3.69}$$

Substituting

$$p = \frac{q}{m^2} \tag{3.70}$$

and acting as in the preceding, we obtain an auxiliary equation of second kind:

$$A_{00}q^{N_0} + A_{01}q^{N_0-1} + \ldots + A_{10}q^{N_0-2} + A_{11}q^{N_0-3} + \ldots \\ \ldots + A_{n0}q^{N_0-2n} + A_{n1}q^{N_0-2n-1} + \ldots \tag{3.71}$$

We have obtained a similar equation before, in our analysis of lagless multivariable systems. It comprises the first two leading coefficients of each polynomial in (3.65).

The system is stable for $m \to 0$ if and only if

(a) the degenerate transcendental equation satisfies the stability conditions,

(b) the auxiliary equation of second kind also satisfies the stability conditions.

Dividing (3.71) through by q^{N_0-2n}, we write the auxiliary equation of second kind in the form

$$A_{00}q^{2n} + A_{01}q^{2n-1} + A_{10}q^{2n-2} + \ldots + A_{n0}q + A_{n1} = 0. \tag{3.72}$$

Finally, if

$$N_0 - N_1 \geqslant 3, \quad N_0 - N_2 \geqslant 6, \quad \ldots, \tag{3.73}$$

the system is unstable for $m \to 0$. The validity of this proposition follows from the property of roots of binomial algebraic equations and is proved in the same way as before.

We thus come to the conclusion that multivariable control systems with time lag which remain stable under unlimited increase of the subsystem gains are realizable. The necessary conditions for this synthesis are specified in the preceding.

Let us elucidate the relationship between the above conditions and the structure of the control system. In other words, we determine the parameters $N_0, N_1, \ldots, N_n$.

Examining the structure of the polynomials in (3.65) and making use of the nomenclature (3.59), we see that the differences $N_0 - N_1$, $N_1 - N_2$, ..., $N_{n-1} - N_n$ are given by the relation

$$N_0 - N_i = m_i + Q_i - n_i, \tag{3.74}$$

where m_i is the degree of the operator p in the denominator of the stabilizer transfer function; Q_i is the degree of the operator p in the denominators of the transfer functions of the elements in the stabilizer section; n_i is the degree of the operator p in the numerator of the stabilizer transfer function.

By assumption

$$N_i - N_{i-1} \leqslant 2,$$

whence

$$m_i - n_i + Q_i \leqslant 2.$$

The degree of the equation describing the stabilized section of systems with infinite-gain stability is thus given by the inequality

$$Q_i \leqslant n_i - m_i + 2. \tag{3.75}$$

An analogous relationship has been derived for single-variable systems and for systems without lag. We thus see that a multivariable control system with time lag remains stable under indefinite increase of gain if and only if each subsystem whose gain is arbitrarily increased belongs to the class of structures with infinite-gain stability.

§ 3.6. MULTIVARIABLE CONTROL SYSTEMS WITH COUPLING THROUGH THE MEASURING DEVICE

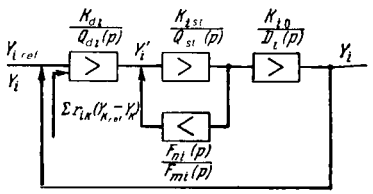

FIGURE 3.3. Block diagram of the i-th loop of a control-coupled system.

Let us consider a particular, but highly significant, class of multivariable systems where the controlled variables are interconnected by the measuring device. Transducer-coupled systems of this kind are generally called multidimensional servosystems. The case of systems consisting of single-loop servos was considered in Chapter Two. We now extend the results of the previous sections of Chapter Three to the case of a multidimensional servosystem block-diagramed in Figure 3.3.

Making use of the nomenclature in Figure 3.3, we write the set of equations describing the dynamics of a multidimensional servosystem in Laplace transforms:

$$\left.\begin{aligned} Q_{di}(p)\,Y_i' &= K_{di}\left\{[Y_{i\,ref}(p) - Y_i(p)] + \sum_{\substack{k=1 \\ k\neq i}}^{n} r_{ik}(p)\,(Y_{k\,ref}(p) - Y_k(p))\right\}, \\ Q_{st\,i}(p)\,Y_i''(p) &= K_{i\,st}\left[Y_i'(p) - \frac{F_{ni}(p)}{F_{mi}(p)}\,Y_i''(p)\right]. \end{aligned}\right\} \tag{3.76}$$

$$D_i(p)Y_i(p)=K_{i0}Y_i''(p)+f_i \tag{3.77}$$

or, eliminating $Y_i'(p)$ and $Y_i''(p)$, we have

$$\begin{aligned}
&\{Q_{di}(p)Q_{st\,i}(p)F_{mi}(p)D_i(p)+K_{i\,st}Q_{di}(p)F_{mi}(p)D_i(p)+\\
&+K_{i\,st}K_{di}K_{i0}F_{mi}(p)\}Y_i(p)+K_{i\,st}K_{di}K_{i0}F_{mi}(p)\sum_{\substack{k=1\\ i\neq k}}^{n}r_{ik}(p)Y_k(p)=\\
&=K_{i\,st}K_{di}K_{i0}F_{mi}(p)\sum_{\substack{k=1\\ k\neq i}}^{n}r_{ik}(p)Y_{i\,ref}(p)+K_{i0}[Q_{di}(p)Q_{st\,i}(p)F_{mi}(p)+\\
&+K_{i\,st}D_{di}(p)F_{mi}(p)]f_i(p)\qquad (k=1,\ 2,\ \ldots,\ n).
\end{aligned} \tag{3.78}$$

The characteristic equation generated by the set (3.78) is

$$\begin{vmatrix}
\beta_{c1}(p)+K_{1\,st}\gamma_{c1}(p)+ & K_{1\,tot}F_{m1}(p)r_{12} & \ldots & K_{1\,tot}F_{m1}(p)r_{1n}\\
+K_{1\,tot}F_{m1}(p) & & & \\
K_{2\,tot}F_{m2}(p)r_{21} & \beta_{c2}(p)+K_{2\,st}\gamma_{c2}(p)+ & \ldots & K_{2\,tot}F_{m2}(p)r_{2n}\\
 & +K_{2\,tot}F_{m2}(p) & & \\
\ldots & \ldots & \ldots & \ldots\\
K_{n\,tot}F_{mn}(p)r_{n1} & \ldots & \ldots\beta_{cn}(p)+K_{n\,st}\gamma_{cn}(p)+ & \\
 & & +K_{n\,tot}F_{mn}(p) &
\end{vmatrix}=0 \tag{3.79}$$

where

$$\left.\begin{aligned}
\beta_{ci}(p)&=Q_{di}(p)Q_{i\,st}(p)F_{mi}(p)D_i(p)\\
\gamma_{ci}(p)&=Q_{i\,st}(p)D_i(p)F_{mi}(p).
\end{aligned}\right\} \tag{3.80}$$

The determinant (3.79) has the same structure as the determinant (3.48), and our results for the synthesis of systems with infinite-gain stability can thus be extended in their entirety to the case of multidimensional servosystems. To be specific, if each component servo considered as a noninteracting system belongs to the class of systems with infinite-gain stability, the entire multidimensional servosystem will remain stable when the subsystem gains are increased indefinitely, provided that the degenerate equations and the auxiliary equations of first, second, and third kinds comply with the stability criteria.

It is easily understood that the results pertaining to the synthesis of stable systems with arbitrarily large loop gains remain valid in the case of systems with simultaneous plant- and transducer-coupling. The same laws also apply when load coupling is additionally introduced. This case, however, is treated in full detail in a separate chapter.

We have thus established the laws of synthesis of multivariable control systems which are stable even though the subsystem gains are increased indefinitely. In the next chapter we will treat on the fundamental properties of these systems.

§ 3.7. DERIVATION OF THE FUNDAMENTAL PROPERTIES OF AUTOMATIC CONTROL SYSTEMS FROM THE *D*-DECOMPOSITION CURVE

In subsequent chapters we will often have to assess the properties of multivariable control systems. The corresponding estimates are

conveniently obtained with the aid of the D-decomposition curve. According to the D-decomposition method, the quality of the system is associated with the numerical values of all the relevant indices. We can actually trace the variation of the system dynamics for various gain values; furthermore, all the estimates are obtained making use of a single D-decomposition curve.

In the beginning let us consider the evaluation of the dynamic properties of single-variable systems. At a later stage, the results will be extended to multivariable systems.

The transfer functions of closed-loop control systems are divided into two groups. The first group includes symmetric transfer functions of the type

$$K_{sy}(p)=\frac{W(p)}{1+W(p)}, \tag{3.81}$$

where $W(p)$ is the transfer function of the open-loop system.

The second group includes asymmetric transfer functions of the form

$$K_{as}(p)=\frac{W_1(p)}{1+W(p)}. \tag{3.82}$$

Here $W_1(p)$ incorporates the external disturbances and is dependent on the point of their application in the system. The initial control conditions can also be incorporated in transfer functions of this general form.

Let the open-loop transfer function be given by a rational-fractional expression

$$W(p)=\frac{R_1(p)}{Q_1(p)}. \tag{3.83}$$

The characteristic equation corresponding to the differential equation of the closed-loop system is then written in the form

$$1+\frac{R_1(p)}{Q_1(p)}=0$$

or

$$R_1(p)+Q_1(p)=0. \tag{3.84}$$

Consider the effect on system dynamics of some parameter τ (the characteristic equation of the system is linear in this parameter):

$$Q(p)+\tau R(p)=0. \tag{3.85}$$

The equation of the D-decomposition curve for the parameter τ has the form

$$\tau=-\frac{Q(j\omega)}{R(j\omega)}. \tag{3.86}$$

The curve plotted using equation (3.86) is a locus of τ-values for which the system remains stable.

The gain-phase characteristic (i.e., Nyquist diagram) of a closed-loop system in the case (3.81) has the form

$$K_{sy}(j\omega)=\frac{W(j\omega)}{1+W(j\omega)} \tag{3.87}$$

or, making use of (3.85) and (3 86),

$$K_{sy}(j\omega) = \frac{\frac{R_1(j\omega)}{R(j\omega)}}{\tau + \frac{Q(j\omega)}{R(j\omega)}}. \qquad (3.88)$$

Equation (3.88) relates the frequency response of a closed-loop control system to the geometry of the D-decomposition curve for the parameter τ.

Let us consider the case when the system gain K is treated as the parameter τ. The gain-phase characteristic of a closed-loop system in the case (3.81) is written in the form

$$K_{s,}(j\omega) = \frac{K}{K + \frac{M(j\omega)}{N(j\omega)}}, \qquad (3.89)$$

where $\frac{M(j\omega)}{N(j\omega)}$ is the equation of the D-decomposition curve for the complex $\bar{K}$.

The quality indices of the system which follow from the properties of the real frequency response are readily obtained from the D-decomposition equation (3.89);* the gain margin, the phase margin and the height of the peak on the closed-loop gain plot are also easily determined using this curve.

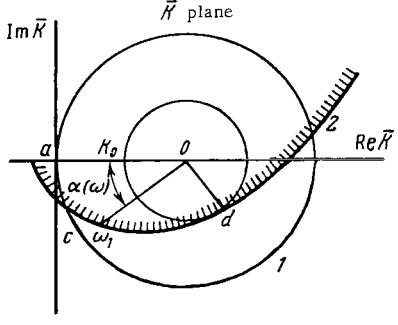

FIGURE 3.4. Derivation of the gain characteristic from D-decomposition curve.

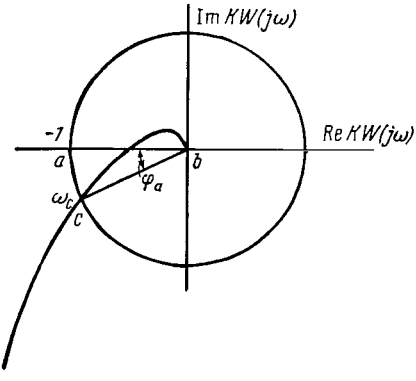

FIGURE 3.5. Estimating phase and gain margin.

Figure 3.4 is a specimen D-decomposition curve for the total gain K. The denominator of (3.89) for some frequency ω_i and a given K_0 (the gain for $\omega = 0$) is determined by the vector bc; the amplitude value of (3.89) for K_0 and ω_i is thus determined by the ratio $\frac{ab}{bc}$. Having found the gain amplitudes for the entire frequency range, we establish the gain characteristic of the system.

* For more details on this subject see Meerov, M. V. Ispol'zovanie krivoi D-razbieniya dlya otsenki kachestva sistem avtomaticheskogo regulirovaniya (D-decomposition Curve for Quality Evaluation of Automatic Control Systems). — Avtomatika i Telemekhanika, 12, No. 6. 1951.

Having selected K_0, we can easily find the peak of the closed-loop gain plot without first constructing the entire response characteristic. Taking K_0 as the center of a circle, we draw a tangent to the D-decomposition curve. The peak of the closed-loop gain plot is then given by K_0 to the radius of the circle, i.e., by the ratio $\frac{ab}{bd}$.

The phase and gain margin can be easily determined from the Nyquist diagram of an open-loop system. The phase margin is found in the following way. A circle of unit radius is drawn around the origin in the gain phase plane (Figure 3.5). The intersection of this circle with the Nyquist plot gives the crossover frequency (or the cutoff frequency) and the angle between the negative real axis and the segment from the origin to the intersection point is the phase margin (angle φ_a in Figure 3.5). In the nomenclature of equation (3.89) the open-loop gain-phase characteristic is expressed by the relation

$$W(j\omega) = \frac{KN(j\omega)}{M(j\omega)}, \tag{3.90}$$

whereas the equation of the D-decomposition curve for $\bar{K}$ is given by

$$\bar{K} = -\frac{M(j\omega)}{N(j\omega)}. \tag{3.91}$$

According to equations (3.90) and (3.91), the phase margin of an open-loop system is determined from the D-decomposition curve in the following way. A circle of radius K_0 is drawn around the origin in the $\bar{K}$ plane; the angle φ_a gives the phase margin (Figure 3.6). The gain margin is obtained without difficulty, since the decomposition curve defines on the $\bar{K}$ plane the entire set of gain values for which the system is stable.

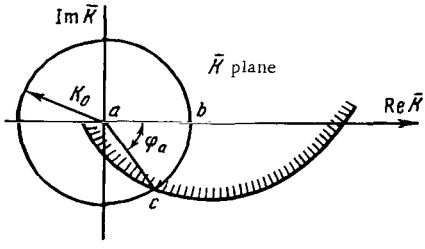

FIGURE 3.6. Estimating phase and gain margin from the D-decomposition curve.

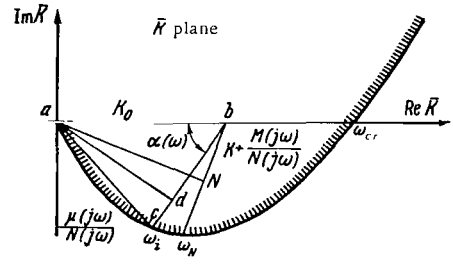

FIGURE 3.7. Construction of the real frequency response from D-decomposition curve.

We now proceed to determine via the D-decomposition curve some quality indices which follow from the properties of the real frequency response of a closed-loop system. First let us show how the closed-loop real frequency response can be obtained from a given D-decomposition curve in the $\bar{K}$ plane (we are concerned with the symmetrical case, see equation (3.81)).

We have already shown how to construct the gain plot of a closed-loop system from the D-decomposition curve in the $\bar{K}$ plane. The real frequency response is obtained without difficulty if, in addition to the gain plot, we can also find the closed-loop phase-angle diagram from the D-decomposition curve.

The phase of $K_{s\gamma}(j\omega)$ for some frequency ω_i is determined by the phase of the denominator in the right-hand side of (3.89) at that frequency. But at the given frequency ω_i the denominator of (3.89) is equal to the segment bc, and the phase of (3.89) at that frequency is $\alpha(\omega_i)$ (see Figure 3.6).

The corresponding phases are thus determined for all the frequencies, and the entire phase-angle plot of a closed-loop system is obtained. The real frequency response $P(\omega)$ is now found without difficulty. At the frequency ω_i we have

$$P(\omega_i) = \frac{ab}{bc} \cos \alpha(\omega_i). \tag{3.92}$$

Dropping a perpendicular from the origin (Figure 3.7) to the segment bc, we obtain from (3.92) the real frequency response at ω_i:

$$P(\omega_i) = \frac{bd}{bc}. \tag{3.93}$$

The value of $P(\omega)$ is obtained by similar geometrical constructions at any frequency, and the entire closed-loop real frequency response is recovered.

Note that the imaginary closed-loop frequency response is also obtained without difficulty; to this end, it suffices to take the segment ratio $\frac{ad}{bc}$ (Figure 3.7).

This method of construction establishes a relationship between the closed-loop real frequency response and the D-decomposition curve. We shall now formulate some quality indices and show how to find them directly from the D-decomposition curve in the $\bar{K}$ plane.

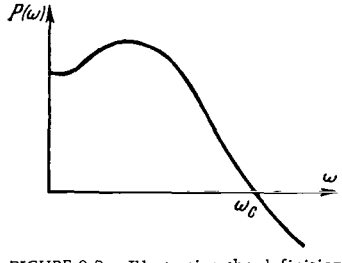

FIGURE 3.8. Illustrating the definition of the positive-response bandwidth.

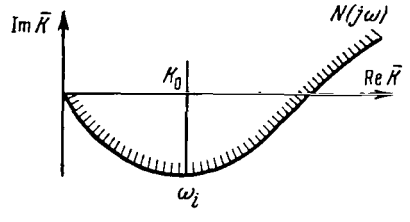

FIGURE 3.9. Estimating the positive-response bandwidth from the D-decomposition curve.

The positive-response bandwidth is defined as the range of frequencies from $\omega = 0$ to the frequency at which the real frequency response crosses the frequency axis for the first time (Figure 3.8). Putting ω_c for this crossover frequency, we write for the control time t

$$t > \frac{\pi}{\omega_c}. \tag{3.94}$$

The positive-response bandwidth is obtained from the D-decomposition curve in the following way. A perpendicular is erected at the point K_0 to its intersection with the D-decomposition curve. The frequency ω_i at the intersection point gives the upper bound of the positiveness range (Figure 3.9).

Other quality indices are similarly obtained from the properties of the real frequency response:

(a) For a time function $x(t)$ to monotonically approach a steady-state value $x(\infty)$, it is necessary (but insufficient) that the D-decomposition curve in the $\bar{K}$ plane does not meet the circle of radius $r=\frac{K_0+1}{2}$ centered at the point $\frac{K_0-1}{2}$ (Figure 3.10).

(b) For the overshoot not to exceed 18% it is necessary that

(1) the magnitude of the vector from the origin to the D-decomposition curve should increase steadily as the frequency increases from 0 to ∞ (Figure 3.11);

(2) for a given K_0 the circle of radius $\frac{K_0+1}{2}$ centered at $\frac{K_0-1}{2}$ should not meet the D-decomposition curve;

(3) the projection of the vector $a\omega_i$ (Figure 3.11) on the K axis should not exceed K_0 for $\omega \to \infty$.

(c) If the initial section of the D-decomposition curve is sufficiently close to an arc of the circle of radius $\frac{K_0+1}{2}$ centered at $\frac{K_0-1}{2}$, the distance between the circle and the D-decomposition curve subsequently increasing (Figure 3.11), the transient time is between the limits

$$\frac{\pi}{\omega_c} < t < \frac{4\pi}{\omega_c}. \tag{3.95}$$

where ω_c is the crossover frequency.

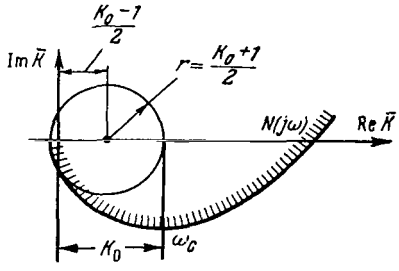

FIGURE 3.10. Determination of necessary conditions for no overshoot from the D- decomposition curve.

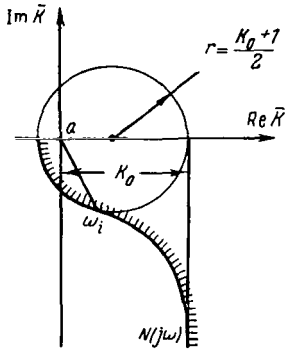

FIGURE 3.11. Determination of quality indices from D-decomposition curve.

It is significant that the above-described properties of the D-decomposition curve are directly related to the magnitude of the total system gain. A number of conclusions can be drawn on the basis of these properties.

The following corollary obtains from property (a) above: if for reasons of precision the total system gain is greater than the diameter (K_0+1) of the circle, the transient process cannot be monotonic. From property (b) we have two corollaries:

(1) To satisfy the sufficient conditions for overshoot not exceeding 18%, the system gain should not be greater than the part of the diameter (K_0+1) to the right of the semiaxis $\mathrm{Re}\,\bar{K}$ (Figure 3.12).

(2) Overshoot will not exceed 18% irrespective of the actual gain if the D-decomposition curve coincides with the imaginary axis in the $\bar{K}$ plane, and the entire positive real axis belongs to the region of stability.

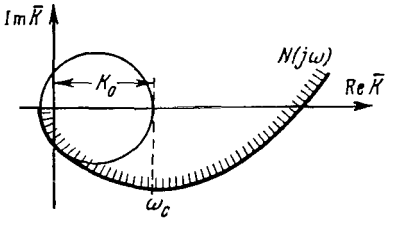

FIGURE 3.12. Illustrating the determination of conditions for overshoot not exceeding 18%.

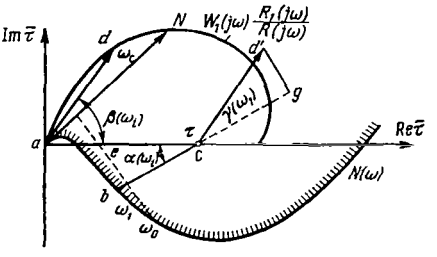

FIGURE 3.13. Illustrating the construction of the closed-loop frequency-response characteristics from D-decomposition curve and auxiliary curve (the auxiliary curve is independent of τ).

A corollary which follows from the method of construction of the positive-response bandwidth has a considerable bearing on the evaluation of the control-system structure. It is easily understood that the upper boundary of the positive-response bandwidth is always less than ω_{cr}, where ω_{cr} is the frequency at the intersection of the D-decomposition curve with the K axis (Figure 3.12) and $K=K_{cr}$ is the critical gain. This corollary will be applied at a later stage to derive some very important conclusions concerning the efficacy of control structures.

We have already emphasized that the above properties of the D-decomposition curve pertain to the case of symmetric transfer functions. Let us now consider some quality indices of a system with an asymmetric transfer function (3.82). First, we construct the real frequency response of the corresponding closed-loop system.

Since external disturbances and initial conditions do not influence the characteristic equation, the general gain-phase characteristic incorporating external disturbances and initial conditions has the form

$$K(j\omega)=\frac{w_1(j\omega)\dfrac{R_1(j\omega)}{R(j\omega)}}{\tau+\dfrac{Q(j\omega)}{R(j\omega)}}. \qquad (3.96)$$

The numerator of (3.96) we call equation of the auxiliary curve. In this more general case, the determination of the system properties is based on the D-decomposition curve and the auxiliary curve. It follows from (3.89) that the auxiliary curve is also required in the symmetric case,

whenever the relevant index is not the gain but some other system parameter. The frequency responses (the real response included) can be easily constructed once the D-decomposition curve for an arbitrary parameter τ and the corresponding auxiliary curve are known. Figure 3.13 is a probable form of a D-decomposition form in the τ plane. It follows from the preceding and directly from Figure 3.13 that the vector $\overline{bc}$ gives the amplitude value of the denominator in (3.96) at the frequency ω_i:

$$\overline{bc} = \tau + \frac{Q(j\omega)}{R(j\omega)}.$$

The phase of the denominator in (3.96) at the frequency ω_i is the angle $\alpha(\omega_i)$.

Let the numerator in (3.96) be independent of τ; a probable auxiliary curve for this case is shown in Figure 3.13. We further assume that the numerator for ω_i is represented by the vector $\overline{ad}$. The phase of the numerator at this frequency is $\beta(\omega_i)$. The amplitude value of (3.96) at the frequency ω_i is given by the ratio of the corresponding segments:

$$\frac{\left| W_1(j\omega) \frac{R_1(j\omega)}{R(j\omega)} \right|}{\left| \tau + \frac{Q(j\omega)}{R(j\omega)} \right|} = \frac{ad}{bc}.$$

The magnitude of (3.96) at any other frequency is obtained similarly, and the entire gain response corresponding to (3.96) is thus recovered.

The phase of (3.96) is the phase of the numerator minus the phase of the denominator. The real frequency response at ω_i is obtained as follows. The vector $\overline{ad}$ is translated from point a to point c (Figure 3.13) and the vector $\overline{bc}$ is continued as is shown in the figure. The phase of (3.96) at the frequency ω_i is then $\gamma(\omega_i)$, since obviously

$$\gamma(\omega_i) = \beta(\omega_i) - \alpha(\omega_i).$$

The real frequency response at the frequency ω_i is

$$\frac{cd}{bc} \cos \gamma(\omega_i).$$

Dropping a perpendicular from the tip of the vector $\overline{cd'}$ to the dashed line, we obtain for the real frequency response at ω_i

$$P(\omega_i) = \frac{cg}{bc}.$$

The real frequency response at any other frequency is obtained in a similar way, so that the entire frequency response of the system is recovered.

If the auxiliary curve is dependent on the parameter τ, we proceed as follows. The numerator in (3.96) is partitioned into two parts, one independent of τ and the other a function of τ. Equation (3.96) is then written as

$$K(j\omega) = \frac{\tau W_2(j\omega) + W_3(j\omega)}{\tau + \frac{Q(j\omega)}{R(j\omega)}}. \qquad (3.97)$$

Figure 3.14 shows the D-decomposition curve in the τ plane and the curve $W_3(j\omega)$. We now choose any particular value of τ, say $\tau = 1$. We can thus find the vector $\tau W_2(j\omega)$ for any frequency ω_i. These vectors are plotted as in Figure 3.14. A choice of any other numerical value for τ only alters the scale of the vector $\tau W_2(j\omega)$. For the given value of τ, the magnitude and the phase of the numerator in (3.97) are represented by the vector joining the origin with the tip of the vector $\tau W_2(j\omega)$ at the corresponding frequency. From this point on, the construction of the frequency-response characteristics proceeds as before, in the case of τ-independent numerator in (3.96).

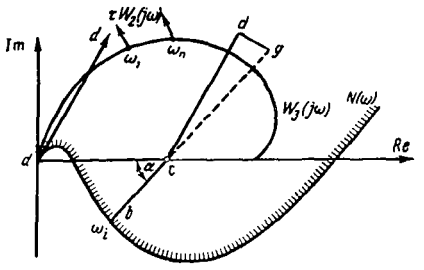

FIGURE 3.14. Illustrating the construction of closed-loop frequency-response characteristics from D-decomposition curve and auxiliary curve (the auxiliary curve is dependent on τ).

The method proposed for the construction of the real frequency response suggests the following properties of the D-decomposition curve and the auxiliary curve, which are useful in the preliminary evaluation of control properties.

A. The positive-response bandwidth is determined by the frequency ω_0 at which the numerator and the denominator vectors assume a mutually perpendicular orientation for the first time. The transient time in this system, as we have already indicated, is

$$t > \frac{\pi}{\omega_c}.$$

If the crossover frequency ω_c is known, the value of τ for which ω_c determines the positive-response bandwidth is found as follows. Draw the numerator vector $a\bar{N}$ at the frequency ω_c (Figure 3.13). From the point ω_c of the D-decomposition curve drop a perpendicular on $a\bar{N}$. The segment ae is the required value of τ.

B. If in some initial frequency range the magnitudes of the numerator and the denominator and the angle $\gamma(\omega)$ between them remain virtually constant, and if subsequently the ratio of the two magnitudes decreases while the angle $\gamma(\omega)$ does not decrease, the control time lies between the limits

$$\frac{\pi}{\omega_c} < t < \frac{4\pi}{\omega_c}.$$

C. The sufficient conditions for overshoot not exceeding 18% are satisfied if condition B is met and the numerator and denominator are not mutually perpendicular at any frequency.

D. The necessary conditions of no overshoot are satisfied if, for respectively equal distribution of frequencies along the numerator and denominator curves, the magnitude of the numerator decreases faster than the magnitude of the denominator at the corresponding frequency.

The above properties of the D-decomposition curve (in the case of a symmetric closed-loop transfer function)and the properties of the D-decomposition curve and the auxiliary curve (in the general case of an asymmetric closed-loop transfer function) will be used in the sequel.

Our estimates can be extended without difficulty to multivariable control systems. At the present stage, we consider the case of a one-parameter D-decomposition curve. In the next chapter it will be shown that the dynamic properties of control systems can be evaluated using the D-decomposition curves for subsystem parameters.

In Chapter Two we derived a general expression for the j-th controlled variable in the most general case of interaction through the plant, the control, and the load. This expression has the form

$$Y_j(p)=\frac{1}{\Delta}\left\{(-1)^{i+j}A_{ij}(p)[K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(p)+g_{ii}(p)f_i(p)]+\right.$$
$$+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(p)\sum_{k=1}^{n}b_{ik}(p)f_k(p)\right]+$$
$$\left.+\sum_{i=1}^{n}\left[(-1)^{i+j}A_{ij}(p)\sum_{\substack{k=1\\k\neq j}}^{n}C_{ik}(p)Y_{k\,\mathrm{ref}}(p)\right]\right\}. \qquad (2.34)$$

The characteristic equation of a multivariable control system is

$$\Delta=0. \qquad (3.98)$$

Suppose that we are concerned with the influence of the parameter τ_i of the i-th subsystem on the dynamic properties of the entire multivariable control system. Note that the parameter τ_i is a linear term in equation (3.98). Under these conditions, equation (3.98) may be written in the form

$$\tau_i\Delta_{ii}+\sum_{\substack{j=1\\j\neq i}}^{n}\Delta_{ij}(-1)^{i+j}=0, \qquad (3.99)$$

whence follows an equation of the D-decomposition curve in the τ_i plane

$$\bar{\tau}_i=-\frac{\sum_{\substack{i=1\\i\neq j}}^{n}\Delta_{ij}(-1)^{i+j}}{\Delta_{ii}}. \qquad (3.100)$$

Dividing (2.34) by $Y_{i\,\mathrm{ref}}(p)$ and making use of (3.49), we write

$$\frac{Y_i(p)}{Y_{i\,\mathrm{ref}}(p)}=\frac{(-1)^{i+j}A_{ij}(p)\left[K_{i\,\mathrm{tot}}+g_{ii}(p)\frac{f_i(p)}{Y_{i\,\mathrm{ref}}(p)}\right]+}{\tau_i\Delta_{ii}+\sum_{\substack{j=1\\j\neq i}}^{n}\Delta_{ij}(-1)^{i+1}}$$

$$+\sum_{i=1}^{n}\left[(-1)^{i+1}A_{ij}(p)\sum_{k=1}^{n}b_{ik}(p)\frac{f_k(p)}{Y_{i\,\mathrm{ref}}(p)}\right]+\sum_{i=1}^{n}\left[(-1)^{i+1}A_{ij}(p)\sum_{\substack{k=1\\k\neq j}}^{n}C_{ik}(p)\frac{Y_{k\,\mathrm{ref}}(p)}{Y_{i\,\mathrm{ref}}(p)}\right]}{\tau_i\Delta_{ii}+\sum_{\substack{j=1\\j\neq i}}^{n}\Delta_{ij}(-1)^{i+1}}. \qquad (3.101)$$

Dividing the numerator and the denominator in (3.101) by Δ_{ii} and putting $p=j\omega$, we write

$$\frac{Y_i(j\omega)}{Y_{i\,\mathrm{ref}}(j\omega)} = \frac{W_{\mathrm{tot}}(j\omega)}{\tau + D_{\tau i}(j\omega)},$$

where

$$W_{\mathrm{tot}}(j\omega) = (-1)^{i+j} A_{ij}(j\omega)\left[K_{i\,\mathrm{tot}} + g_{ii}(j\omega)\frac{f_i(j\omega)}{Y_{i\,\mathrm{ref}}(j\omega)}\right] +$$

$$+ \sum\left[(-1)^{i+j} A_{ij}(j\omega)\sum_{k=1}^{n} b_{ik}(j\omega)\frac{f_k(j\omega)}{Y_{i\,\mathrm{ref}}(j\omega)}\right] + \sum\left[(-1)^{i+j} A_{ij}(j\omega)\sum_{\substack{k=1\\k\neq j}}^{n} C_{ik}(j\omega)\frac{Y_{k\,\mathrm{ref}}(j\omega)}{Y_{k\,\mathrm{ref}}(j\omega)}\right]$$

is in fact the equation of the auxiliary curve. We easily see that

$$D_{\tau i}(j\omega) = \frac{\sum_{l=1,\ l\neq j}^{n} \Delta_{lj}(-1)^{l-j}}{\Delta}$$

is the equation of the D-decomposition curve (apart from the sign).

All the previous results concerning the application of D-decomposition and auxiliary curves for the evaluation of dynamic properties of single-variable control systems can thus be extended to multivariable controls.

Chapter Four

GENERAL PROPERTIES OF MULTIVARIABLE CONTROL SYSTEMS WITH INFINITE-GAIN STABILITY

§ 4.1. DERIVATION OF THE GENERAL EQUATION

(a) PROPORTIONAL SYSTEMS

In the previous chapter we established general rules for the design of multivariable control systems which permit indefinitely increasing the gain of the various subsystems without losing their stability as a whole. The fundamental properties of these infinite-gain stable systems can be determined by examining their matrix equation.

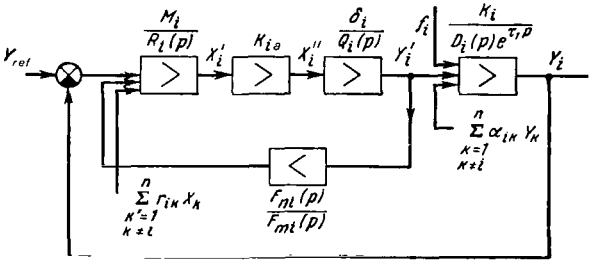

FIGURE 4.1. Illustrating the derivation of multivariable control equation: proportional systems.

Suppose that the controlled variables are coupled through the plant and the measurement devices. Figure 4.1 is a block diagram of the prototype system analyzed in this section. Stabilization is provided by an elastic negative feedback element connecting the plant output with the input of the measuring device. Alternative feedback configurations will be considered in what follows. The essential point is that this system belongs to the class of structures with infinite-gain stability.

It follows from the results of the preceding chapter that the system depicted in Figure 4.1 must satisfy the following structural conditions:

1) The polynomial $\alpha_{ik}(p)\alpha_{ki}(p)$ is of lower degree than the polynomial

$$D_i(p)D_k(p). \tag{4.1}$$

2) $$n_i - m_i + r_i + q_i \leqslant 2, \tag{4.2}$$

where n_i, m_i, r_i, q_i are the degrees of the polynomials $F_{ni}(p)$, $F_{mi}(p)$, $R_i(p)$ and $Q(p)$, respectively.

We now derive the equation of the system in Laplace transforms.

A. The equation of the controlled object

$$D_i(p)\,e^{\tau_i p}Y_i(p)=K_i\left[Y_i'(p)-\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\,Y_k(p)+f_i(p)\right]. \tag{4.3}$$

B. The equation of the measurement device

$$R_i(p)\,X_i'=\mu_i\left[Y_{i\,\mathrm{ref}}(p)-Y_i(p)+\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)\,(Y_{k\,\mathrm{ref}}(p)-Y_k(p))-\frac{F_{ni}(p)}{F_{mi}(p)}\,Y_i'(p)\right]. \tag{4.4}$$

C. The amplifier equation

$$X_i''(p)=K_{i\,a}X_i'(p). \tag{4.5}$$

D. The controller equation

$$Q_i(p)\,Y_i'(p)=\delta_i X_i''(p). \tag{4.6}$$

Eliminating $Y_i'(p)$, $X_i'(p)$ and $X_i''(p)$ between equations (4.3)–(4.6), we obtain after simple manipulations

$$\begin{aligned}
&[D_i(p)\,R_i(p)\,Q_i(p)\,F_{mi}(p)\,e^{\tau_i p}+\mu_i\delta_i K_{i\,a}D_i(p)\,F_{ni}(p)\,e^{\tau_i p}+\\
&+K_iK_{i\,a}\delta_i\mu_i F_{mi}(p)]\,Y_i(p)+\left[K_i\mu_iK_{i\,a}\delta_iF_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)+\right.\\
&\left.+K_iR_i(p)\,Q_i(p)\,F_{ni}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\right]Y_k(p)+\\
&+K_i\mu_i\delta_iK_{i\,a}F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\,Y_k(p)=K_iK_{i\,a}\mu_i\delta_iF_{mi}(p)\,Y_{i\,\mathrm{ref}}(p)+\\
&+K_iK_{i\,a}\mu_i\delta_iF_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)\,Y_{i\,\mathrm{ref}}(p)+K_iR_i(p)\,Q_i(p)F_{mi}(p)f_i(p)+\\
&+K_i\mu_i\delta_iK_{i\,a}F_{ni}(p)\,f_i(p)\qquad(i=1,\,2,\,\ldots,\,n).
\end{aligned} \tag{4.7}$$

We introduce the following notation:

$$\left.\begin{aligned}
&D_i(p)\,R_i(p)\,Q_i(p)\,F_{mi}(p)\,e^{\tau_i p}=a_i(p);\ \mu_i\delta_iK_{i\,a}=K_{i\,\mathrm{st}},\\
&D_i(p)\,F_{ni}(p)\,e^{\tau_i p}=b_i(p);\quad K_iK_{i\,a}\delta_i\mu_i=K_{i\,\mathrm{tot}},\\
&K_iR_i(p)\,Q_i(p)\,F_{mi}(p)=c_i(p);\ a_i(p)+K_{i\,\mathrm{st}}\,b_i(p)+\\
&\qquad\qquad+K_{i\,\mathrm{tot}}F_{mi}(p)=a_{ii}(p).
\end{aligned}\right\} \tag{4.8}$$

Now equation (4.7) takes the form

$$a_{ii}(p)Y_i(p)+\left[K_{i\,\mathrm{tot}}F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)+c_i\sum_{k=1}^{n}a_{ik}(p)+\right.$$

$$\left.+K_{i\,\mathrm{tot}}F_{ni}(p)\sum_{\substack{k=1\\k\neq i}}^{n}a_{ik}(p)\right]Y_k(p)=K_{i\,\mathrm{tot}}F_{mi}(p)Y_{i\,\mathrm{ref}}(p)+$$

$$+K_{i\,\mathrm{tot}}F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_{i\,\mathrm{ref}}(p)+c_if_i(p)+K_{i\,\mathrm{tot}}F_{ni}(p)f_i(p). \tag{4.9}$$

The complete set of equations is obtained by putting $i = 1, 2, \ldots$

The complete set of equations can be written in matrix form:

$$AY=(K_{\mathrm{tot}}F_m+B)Y_{\mathrm{ref}}+NF. \tag{4.10}$$

Here

$$A=\begin{Vmatrix} a_1(p)+K_{1\,\mathrm{st}}\,b_1(p)+K_{1\,\mathrm{tot}}F_{m1}(p) & K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{12}(p)+K_{1\,\mathrm{tot}}F_{n1}(p)+C_1(p)a_{12}(p) & \ldots & K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{1n}(p)+K_{1\,\mathrm{tot}}F_{n1}(p)+C_1(p)\,a_{1n}(p) \\ \ldots & \ldots & \ldots & \ldots \\ K_{i\,\mathrm{tot}}F_{mi}(p)\,r_{i1}(p)+K_{i\,\mathrm{tot}}F_{ni}(p)+C_i(p))\,a_{i1}(p) & a_i(p)+K_{i\,\mathrm{st}}\,b_i(p)+K_{i\,\mathrm{tot}}F_{mi}(p) & \ldots & K_{i\,\mathrm{tot}}F_{mi}(p)\,r_{in}(p)+(K_{i\,\mathrm{tot}}F_{ni}(p)+C_i(p))\,a_{in}(p) \\ \ldots & \ldots & \ldots & \ldots \\ K_{n\,\mathrm{tot}}F_{mn}(p)\,r_{n1}(p)+(K_{n\,\mathrm{tot}}F_{nn}(p)+C_n(p))\,a_{n1}(p) & \ldots & \ldots & a_n(p)+K_{n\,\mathrm{tot}}b_n(p)+K_{n\,\mathrm{tot}}F_{mn}(p) \end{Vmatrix},$$

$$K_{\mathrm{tot}}F_m=\begin{Vmatrix} K_{1\,\mathrm{tot}}F_{m1}(p) & 0 & \ldots & 0 \\ 0 & K_{2\,\mathrm{tot}}F_{m2}(p) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & K_{n\,\mathrm{tot}}F_{mn}(p) \end{Vmatrix}, \tag{4.11}$$

$$B=\begin{Vmatrix} 0 & K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{12}(p) & \ldots & K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{1n}(p) \\ K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p) & 0 & \ldots & K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{2n}(p) \\ \ldots & \ldots & \ldots & \ldots \\ K_{n\,\mathrm{tot}}F_{mn}(p)\,r_{n1}(p) & \ldots & \ldots & 0 \end{Vmatrix},$$

$$N=\begin{Vmatrix} c_1(p)+K_{1\,\mathrm{tot}}F_{n1}(p) & 0 & \ldots & 0 \\ 0 & c_2(p)+K_{2\,\mathrm{tot}}F_{n2}(p) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & c_n(p)+K_{n\,\mathrm{tot}}F_{nn}(p) \end{Vmatrix},$$

$$Y=\begin{Vmatrix} Y_1(p) \\ Y_2(p) \\ \vdots \\ Y_n(p) \end{Vmatrix},\quad Y_{\mathrm{ref}}=\begin{Vmatrix} Y_{1\,\mathrm{ref}}(p) \\ Y_{2\,\mathrm{ref}}(p) \\ \vdots \\ Y_{n\,\mathrm{ref}}(p) \end{Vmatrix},\quad F=\begin{Vmatrix} f_1(p) \\ f_2(p) \\ \vdots \\ f_n(p) \end{Vmatrix}.$$

From (4.10) we obtain a general matrix equation for the vector value of the controlled variables:

$$Y=A^{-1}[(K_{\mathrm{tot}}F_m+B)Y_{\mathrm{ref}}+NF]. \tag{4.12}$$

Further analysis requires explicit expressions for each i-th variable. This can be done along the same lines as in Chapter Two, where a simpler case was considered.

The system determinant Δ is

$$\Delta = \begin{vmatrix} a_1(p)+K_{1\,st}\ b_1(p)+ & K_{1tot}F_{m1}(p)\,r_{12}(p)+ & \dots & K_{1tot}F_{m1}(p)\,r_{1n}(p)+ \\ +K_{1tot}F_{m1}(p) & +(K_{1tot}F_{n1}(p)+ & & +(K_{1tot}F_{n1}(p)+ \\ & +C_1(p))\,\alpha_{12}(p) & & +C_1(p))\,\alpha_{1n}(p) \\ K_{2tot}F_{m2}(p)\,r_{21}(p)+ & a_2(p)+K_{2\,st}\ b_2(p)+ & \dots & K_{2tot}F_{m2}(p)\,r_{2n}(p)+ \\ +(K_{2tot}F_{n2}(p)+ & +K_{2tot}F_{m2}(p) & & +K_{2tot}F_{n2}(p)+ \\ +C_2(p))\,\alpha_{21}(p) & & & +C_2(p))\,\alpha_{2n}(p) \\ \dots & \dots & \dots & \dots \\ K_{ntot}F_{mn}(p)\,r_{n1}(p)+ & \dots & \dots & a_n(p)+K_{n\,st}\ b_n(p)+ \\ +K_{ntot}F_{nn}(p)+ & & & +K_{ntot}F_{mn}(p) \\ +C_n(p)\,\alpha_{n1}(p) & & & \end{vmatrix}. \tag{4.13}$$

The transpose of matrix (4.11) is

$$A_t = \begin{vmatrix} a_1(p)+K_{1\,st}\ b_1(p)+ & K_{2tot}F_{m2}(p)\,r_{21}(p)+ & \dots & K_{ntot}F_{mn}(p)\,r_{n1}(p)+ \\ +K_{1tot}F_{m1}(p) & +(K_{2tot}F_{n2}(p)+ & & +K_{ntot}F_{nn}(p)+ \\ & +C_2(p))\,\alpha_{21}(p) & & +C_n(p)\,\alpha_{n1}(p) \\ K_{1tot}F_{m1}(p)\,r_{12}(p)+ & a_2(p)+K_{2\,st}\ b_2(p)+ & \dots & K_{ntot}F_{mn}(p)\,r_{n2}(p)+ \\ +(K_{1tot}F_{n1}(p)+ & +K_{2tot}F_{m2}(p) & & +C_n(p)\,\alpha_{n2}(p) \\ +C_1(p))\,\alpha_{12}(p) & & & \\ \dots & \dots & \dots & \dots \\ K_{1tot}F_{m1}(p)\,r_{1n}(p)+ & \dots & \dots & a_n(p)+K_{n\,st}\ b_n(p)+ \\ +(K_{1tot}F_{n1}(p)+ & & & +K_{ntot}F_{mn}(p) \\ +C_1((p))\,\alpha_{1n}(p) & & & \end{vmatrix}. \tag{4.14}$$

Making use of (4.13) and (4.14) and remembering how the inverse of a matrix is formed, we write

$$A^{-1} = -\frac{1}{\Delta}\begin{vmatrix} A_{11} & (-1)^{i+1}A_{i1} & \dots & (-1)^{n+1}A_{n1} \\ -A_{12} & (-1)^{i+2}A_{i2} & \dots & (-1)^{n+2}A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1j}(-1)^{j+1} & (-1)^{j+i}A_{ij} & \dots & (-1)^{n+j}A_{nj} \\ \dots & \dots & \dots & \dots \\ A_{1n}(-1)^{n+1} & (-1)^{i+n}A_{in} & \dots & A_{nn} \end{vmatrix}, \tag{4.15}$$

where $(-1)^{j+i}A_{ij}$ are the cofactors of the elements of the transpose. Carrying out the multiplication in the right-hand side of (4.10) and making use of (4.12), (4.13), (4.14), and (4.15), we find

$$\begin{Vmatrix} Y_1(p) \\ Y_2(p) \\ \vdots \\ Y_n(p) \end{Vmatrix} = \frac{1}{\Delta}\begin{vmatrix} A_{11} & -A_{21} & \dots & (-1)^{i+1}A_{i1} & \dots & (-1)^{n+1}A_{n1} \\ -A_{12} & A_{22} & & (-1)^{i+2}A_{i2} & \dots & (-1)^{n+2}A_{n2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (-1)^{1+j}A_{1j} & & \dots & (-1)^{i+j}A_{ij} & \dots & (-1)^{n+j}A_{nj} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (-1)^{1+n}A_{1n} & & \dots & (-1)^{i+n}A_{in} & \dots & A_{nn} \end{vmatrix} \times$$

$$\times \begin{vmatrix} K_{1tot}F_{m1}(p)\,Y_{1ref}(p)+K_{1tot}F_{m1}(p)\sum_{2}^{n} r_{1k}(p)\,Y_{k\,ref}(p)+ \\ \qquad +(c_1(p)+K_{1tot}F_{n1}(p))\,f_1(p) \\ K_{2tot}F_{m2}(p)\,Y_{2ref}(p)+K_{2tot}F_{m2}(p)\sum_{\substack{k=1 \\ k\neq 2}}^{n} r_{2k}(p)\,Y_{k\,ref}(p)+ \\ \qquad +(c_2(p)+K_{2tot}F_{n2}(p))\,f_2(p) \\ \dots \\ K_{jtot}F_{mj}(p)\,Y_{jref}(p)+K_{jtot}K_{mj}(p)\sum_{\substack{k=1 \\ k\neq j}}^{n} r_{jk}(p)\,Y_{k\,ref}(p)+ \\ \qquad +(c_j(p)+K_{jtot}F_{nj}(p))\,f_j(p) \\ \dots \\ K_{ntot}F_{mn}(p)\,Y_{nref}(p)+K_{ntot}F_{mn}(p)\sum_{k=1}^{n-1} r_{nk}(p)\,Y_{k\,ref}(p)+ \\ \qquad +(c_n(p)+K_{ntot}F_{mn}(p))\,f_n(p) \end{vmatrix} \tag{4.16}$$

Performing the matrix multiplication in the right-hand side of (4.16), we find

$$\begin{Vmatrix} Y_1(p) \\ Y_2(p) \\ \vdots \\ Y_j(p) \\ \vdots \\ Y_n(p) \end{Vmatrix} = \frac{1}{\Delta} \begin{Vmatrix} \sum_{i=1}^{n} (-1)^{1+i} A_{i1} \left[K_{i\,\mathrm{tot}} F_{mi}(p)\, Y_{i\,\mathrm{ref}}(p) + \right. \\ \left. + K_{i\,\mathrm{tot}} F_{mi}(p) \sum_{k=2}^{n} r_{ik}(p)\, Y_{k\,\mathrm{ref}}(p) + (c_i(p) + K_{i\,\mathrm{tot}} F_{ni}(p)) f_i(p) \right] \\ \sum_{i=1}^{n} (-1)^{2+i} A_{i2} \left[K_{i\,\mathrm{tot}} F_{mi}(p)\, Y_{i\,\mathrm{ref}}(p) + \right. \\ \left. + K_{i\,\mathrm{tot}} F_{mi}(p) \sum_{\substack{k=1 \\ k \neq 2}}^{n} r_{ik}(p)\, Y_{k\,\mathrm{ref}}(p) + (c_i(p) + K_{i\,\mathrm{tot}} F_{ni}(p)) f_i(p) \right] \\ \cdots\cdots\cdots\

where

$$\Delta_0 = \begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}}r_{12}+\alpha_{12} & \dots & K_{1\,\mathrm{tot}}r_{1n}+\alpha_{1n} \\ K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} & 1+K_{2\,\mathrm{tot}} & \dots & K_{2\,\mathrm{tot}}r_{2n}+\alpha_{2n} \\ \dots & \dots & \dots & \dots \\ K_{n\,\mathrm{tot}}r_{n1}+\alpha_{n1} & \dots & \dots & 1+K_{n\,\mathrm{tot}} \end{Vmatrix}, \tag{4.21}$$

$$A_{i_0} = \begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} & \dots & K_{n\,\mathrm{tot}}r_{n1}+\alpha_{n1} \\ K_{1\,\mathrm{tot}}r_{12}+\alpha_{12} & 1+K_{2\,\mathrm{tot}} & \dots & K_{n\,\mathrm{tot}}r_{n2}+\alpha_{n2} \\ \dots & \dots & \dots & \dots \\ K_{1\,\mathrm{tot}}r_{1j}+\alpha_{1j} & \dots & 1+K_{j\,\mathrm{tot}} & K_{n\,\mathrm{tot}}r_{nj}+\alpha_{nj} \\ \dots & \dots & \dots & \dots \\ K_{1\,\mathrm{tot}}r_{1n}+\alpha_{1n} & \dots & \dots & 1+K_{n\,\mathrm{tot}} \end{Vmatrix}. \tag{4.22}$$

All $A_{ij}(0)$ can be found from (4.22).

The steady-state value of any j-th variable can be obtained from (4.20). As an example, let us consider a system with three interrelated controlled variables. To find the steady-state value of the second, we write

$$Y_2(0) = \frac{1}{\Delta_{03}} \sum_{i=1}^{3} (-1)^{2+i} A_{2i}(0) \left[K_{i\,\mathrm{tot}} Y_{i\,\mathrm{ref}}(0) + K_{i\,\mathrm{tot}} \sum_{\substack{k=1 \\ k\neq 2}}^{3} r_{2k} Y_{k\,\mathrm{ref}}(0) + K_i f_i \right], \tag{4.23}$$

$$\begin{aligned} \Delta_{03} &= \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}}r_{12}+\alpha_{12} & K_{1\,\mathrm{tot}}r_{13}+\alpha_{13} \\ K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} & 1+K_{2\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{23}+\alpha_{23} \\ K_{3\,\mathrm{tot}}r_{31}+\alpha_{31} & K_{3\,\mathrm{tot}}r_{32}+\alpha_{32} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \\ A_{i_0} &= \begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} & K_{3\,\mathrm{tot}}r_{31}+\alpha_{31} \\ K_{1\,\mathrm{tot}}r_{12}+\alpha_{12} & 1+K_{2\,\mathrm{tot}} & K_{3\,\mathrm{tot}}r_{32}+\alpha_{32} \\ K_{1\,\mathrm{tot}}r_{13}+\alpha_{13} & K_{2\,\mathrm{tot}}r_{23}+\alpha_{23} & 1+K_{3\,\mathrm{tot}} \end{Vmatrix}, \end{aligned} \tag{4.24}$$

whence

$$A_{21} = \begin{vmatrix} K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} & K_{3\,\mathrm{tot}}r_{31}+\alpha_{31} \\ K_{2\,\mathrm{tot}}r_{23}+\alpha_{23} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \tag{4.25}$$

$$A_{22} = \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_{3\,\mathrm{tot}}r_{31}+\alpha_{31} \\ K_{1\,\mathrm{tot}}r_{13}+\alpha_{13} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \tag{4.26}$$

$$A_{23} = \begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{21}+\alpha_{21} \\ K_{1\,\mathrm{tot}}r_{13}+\alpha_{13} & K_{2\,\mathrm{tot}}r_{23}+\alpha_{23} \end{vmatrix}. \tag{4.27}$$

Substituting (4.27) in (4.25), we obtain after simple manipulations

$$\begin{aligned} Y_2(0) = \frac{1}{\Delta_{03}} \{ &- [(K_{2\,\mathrm{tot}}r_{21}+\alpha_{21})(1+K_{3\,\mathrm{tot}}) - \\ &- (K_{2\,\mathrm{tot}}r_{23}+\alpha_{23})(K_{3\,\mathrm{tot}}r_{31}+\alpha_{31})] \times \\ &\times [K_{1\,\mathrm{tot}}Y_{i\,\mathrm{ref}}(0) + K_{1\,\mathrm{tot}}(r_{21}Y_{1\,\mathrm{ref}}(0) + r_{23}Y_{3\,\mathrm{ref}}) + K_1 f_1] + \\ &+ [(1+K_{1\,\mathrm{tot}})(1+K_{3\,\mathrm{tot}}) - (K_{1\,\mathrm{tot}}r_{13}+\alpha_{13})(K_{3\,\mathrm{tot}}r_{31}+\alpha_{31})] \times \\ &\times [K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}(0) + K_{2\,\mathrm{tot}}(r_{21}Y_{1\,\mathrm{ref}}(0) + r_{23}Y_{3\,\mathrm{ref}}(0))K_2 f_2] - \\ &- [(1+K_{1\,\mathrm{tot}})(K_{2\,\mathrm{tot}}r_{23}+\alpha_{23})(K_{1\,\mathrm{tot}}r_{13}+\alpha_{13})(K_{2\,\mathrm{tot}}r_{21}+\alpha_{23})] \times \\ &\times [K_{3\,\mathrm{tot}}Y_{3\,\mathrm{ref}}(0) + K_{3\,\mathrm{tot}}(r_{21}Y_{1\,\mathrm{ref}}(0) + r_{23}Y_{3\,\mathrm{ref}}(0)) + K_3 f_3]\}. \end{aligned} \tag{4.28}$$

The steady-state value can now be calculated if the numerical values of all the parameters are known. Furthermore, some general properties of these systems under steady-state conditions can be established. An interesting particular case is provided by an ordinary plant-coupled multivariable system having $r_{ik} = 0$ and by a multidimensional servo-system with $\alpha_{ik} = 0$.

Let us first consider an ordinary multivariable system ($r_{ik}=0$). Putting $r_{ik}=0$ in (4.28) and (4.24), we find

$$Y_2(0)=\frac{1}{\begin{vmatrix} 1+K_{1\,\mathrm{tot}} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 1+K_{2\,\mathrm{tot}} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}}\{-[(1+K_{3\,\mathrm{tot}})\alpha_{21}-\alpha_{23}\alpha_{31}]\times$$
$$\times[K_{1\,\mathrm{tot}}Y_{1\,\mathrm{ref}}(0)+K_1f_1]+[(1+K_{1\,\mathrm{tot}})(1+K_{3\,\mathrm{tot}})-\alpha_{13}\alpha_{31}]\times$$
$$\times[K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}(0)+K_2f_2]-[(1+K_{1\,\mathrm{tot}})\alpha_{23}-\alpha_{13}\alpha_{23}]\times$$
$$\times[K_{3\,\mathrm{tot}}Y_{3\,\mathrm{ref}}(0)+K_3f_3]\}. \qquad (4.29)$$

Dividing the numerator and the denominator in (4.29) by $K_{1\,\mathrm{tot}}K_{2\,\mathrm{tot}}K_{3\,\mathrm{tot}}$ and taking $K_{1\,\mathrm{tot}}=K_{2\,\mathrm{tot}}=K_{3\,\mathrm{tot}}\to\infty$, we find

$$\lim_{K_{\mathrm{tot}}\to\infty} Y_2(0)=Y_{2\,\mathrm{ref}}(0). \qquad (4.30)$$

It is clear from (4.30) that the accuracy of each controlled variable increases as all the subsystem gains are increased. In the limit $Y_i=Y_{i\,\mathrm{ref}}$: moreover, the coupling between the individual controlled variables vanishes in the limit and they become independent, noninteracting. This result derived for the particular case of a three-variable system is readily generalized to any multivariable control system. Indeed, in the nonsingular case the rank of the determinant A_{ij} is one less than the rank of the determinant Δ_{0n} and the maximum number of factors $K_{k\,\mathrm{tot}}$ in its expansion is $\prod_{\substack{k=1\\k\neq i}}^{n} K_{k\,\mathrm{tot}}$. This product is further multiplied by $K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}$ so that

$$\lim_{\prod_{i=1}^{n} K_{i\,\mathrm{tot}}\to\infty} Y_j(0)\to Y_{j\,\mathrm{ref}}. \qquad (4.31)$$

Increasing all the subsystem gain parameters (which in this case is structurally permissible without loss of stability) thus ensures that the controlled variable retains its steady-state value to arbitrarily high accuracy and that the j-th controlled variable is independent of all the rest. If the gain is high but finite, the steady-state accuracy is not ideal, and uncoupling is achieved to accuracy of ε (we shall refer to it as ε-uncoupling). The value of ε can be determined if the numerical values of all the parameters in the equation are known.

As an example let us determine the steady-state value $Y_1(0)$ in the particular case of a two-variable control system with the following parameters:

$$K_1=500,\ K_2=500,\ \alpha_{12}=\alpha_{21}=0.5,\ f_1=1,\ f_2=1,\ K_1=2,\ K_2=3.$$

From (4.20) for $n=2$ we have

$$Y_1(0)=\frac{1}{\Delta_{02}}\left[\sum_{i=1}^{2}(-1)^{1+i}A_{1i}(0)\right][K_{i\,\mathrm{tot}}Y_{i\,\mathrm{ref}}+K_if_i]. \qquad (4.32)$$

Substituting, we find

$$\Delta_{02}=\begin{vmatrix} 1+K_{1\,\mathrm{tot}} & \alpha_{12} \\ \alpha_{21} & 1+K_{2\,\mathrm{tot}} \end{vmatrix}=(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})-\alpha_{12}\alpha_{21},$$

$$A_{t_{02}}=\begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & \alpha_{21} \\ \alpha_{12} & 1+K_{2\,\mathrm{tot}} \end{Vmatrix},$$

$$A_{11}=1+K_{2\,\mathrm{tot}},\quad A_{12}=\alpha_{12},$$

$$Y_{10}(0)=\frac{1}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})-\alpha_{12}\alpha_{21}}\{(1+K_{2\,\mathrm{tot}})(K_{1\,\mathrm{tot}}Y_{1\,\mathrm{ref}}+K_1f_1)-$$
$$-\alpha_{21}(K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}(0)+K_2f_2)\}=$$
$$=\frac{(1+K_{2\,\mathrm{tot}})(K_{1\,\mathrm{tot}}Y_{1\,\mathrm{ref}}+K_1f_1-\alpha_{21}K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}(0)-\alpha_{21}K_2f_2)}{(1+K_{1\,\mathrm{tot}})(1+K_{2\,\mathrm{tot}})-\alpha_{12}\alpha_{21}}=$$
$$=\frac{250\,500Y_{1\,\mathrm{ref}}(0)+1000f_1-0.5\cdot 500Y_{2\,\mathrm{ref}}(0)-1.5f_2}{25\,100-0.25}=$$
$$=0.99Y_{1\,\mathrm{ref}}(0)+\frac{1}{251}f_1-\frac{25}{25\,100}Y_{2\,\mathrm{ref}}(0)-\frac{1.5}{25\,100}f_2.$$

It follows that already for $K_{\mathrm{tot}}=500$ the effect of extraneous parameters (i.e., the effect of coupling) in plant-coupled multivariable control system is vanishingly small under steady-state conditions.

Let us now consider the case of a three-dimensional servosystem. All α_{ik} are zero, and the determinant is

$$\Delta=\begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}}r_{12} & K_{1\,\mathrm{tot}}r_{13} \\ K_{2\,\mathrm{tot}}r_{21} & 1+K_{2\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{23} \\ K_{3\,\mathrm{tot}}r_{31} & K_{3\,\mathrm{tot}}r_{32} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \tag{4.33}$$

$$A_t=\begin{Vmatrix} 1+K_{1\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{21} & K_{3\,\mathrm{tot}}r_{31} \\ K_{1\,\mathrm{tot}}r_{12} & 1+K_{2\,\mathrm{tot}} & K_{3\,\mathrm{tot}}r_{32} \\ K_{1\,\mathrm{tot}}r_{13} & K_{2\,\mathrm{tot}}r_{23} & 1+K_{3\,\mathrm{tot}} \end{Vmatrix}. \tag{4.34}$$

The properties of the controlled variables can be elucidated for the particular case of $Y_1(0)$. A_{11}, A_{21} and A_{31} are required for the analysis, and from (4.34) we have with proper signs

$$A_{11}=\begin{vmatrix} 1+K_{2\,\mathrm{tot}} & K_{3\,\mathrm{tot}}r_{32} \\ K_{2\,\mathrm{tot}}r_{33} & 1+K_{3\,\mathrm{tot}} \end{vmatrix},$$
$$A_{21}=-\begin{vmatrix} K_{1\,\mathrm{tot}}r_{12} & K_{3\,\mathrm{tot}}r_{32} \\ K_{1\,\mathrm{tot}}r_{13} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}, \tag{4.35}$$
$$A_{31}=\begin{vmatrix} K_{1\,\mathrm{tot}}r_{12} & 1+K_{2\,\mathrm{tot}} \\ K_{1\,\mathrm{tot}}r_{13} & K_{2\,\mathrm{tot}}r_{33} \end{vmatrix},$$

$$Y_1(0)=\frac{1}{\begin{vmatrix} 1+K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}}r_{12} & K_{1\,\mathrm{tot}}r_{13} \\ K_{2\,\mathrm{tot}}r_{21} & 1+K_{2\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{23} \\ K_{3\,\mathrm{tot}}r_{31} & K_{3\,\mathrm{tot}}r_{32} & 1+K_{3\,\mathrm{tot}} \end{vmatrix}}\times$$
$$\times\{[(1+K_{2\,\mathrm{tot}})(1+K_{3\,\mathrm{tot}})-K_{2\,\mathrm{tot}}K_{3\,\mathrm{tot}}r_{23}r_{32}][K_{1\,\mathrm{tot}}Y_{1\,\mathrm{ref}}+K_{1\,\mathrm{tot}}\times$$
$$\times(r_{12}Y_{2\,\mathrm{ref}}+r_{13}Y_{3\,\mathrm{ref}})+K_1f_1]-[K_{1\,\mathrm{tot}}r_{12}(1+K_{3\,\mathrm{tot}})-K_{1\,\mathrm{tot}}K_{3\,\mathrm{tot}}r_{13}r_{32}]\times$$
$$\times[K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}+K_{2\,\mathrm{tot}}(r_{21}Y_{1\,\mathrm{ref}}+r_{23}Y_{3\,\mathrm{ref}})+K_2f_2]+$$
$$+[K_{1\,\mathrm{tot}}r_{12}K_{2\,\mathrm{tot}}r_{23}-K_{1\,\mathrm{tot}}r_{13}(1+K_{2\,\mathrm{tot}})]\times$$
$$\times[K_{3\,\mathrm{tot}}Y_{3\,\mathrm{ref}}+K_{3\,\mathrm{tot}}(r_{31}Y_{1\,\mathrm{ref}}+r_{32}Y_{2\,\mathrm{ref}})+K_3f_3]\}. \tag{4.35'}$$

$Y_1(0)$ can be found from (4.35') if the numerical values of all the parameters are known. The degree of influence of the other inputs (the extraneous reference values) depends on r_{ik}. It is also clear from (4.35') that increasing the subsystem gains does not decouple the system.

(b) INTEGRAL SYSTEMS

Let us now consider a structure with integrating (floating) control. In Figure 4.2 the stabilizer embraces only part of the forward path, which does not include the measuring device. For integral control it is necessary and sufficient that the self-operator $R'(p)$ contain an integrating element, i.e., $R'(p)=pR(p)$. We now write the equations describing the transient and steady-state properties of this configuration.

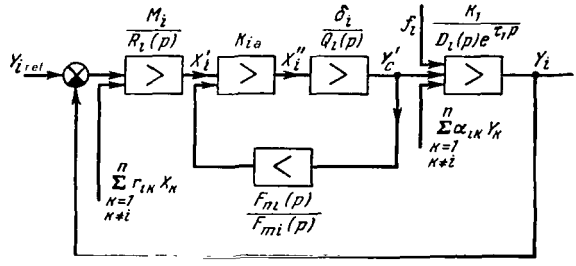

FIGURE 4.2. Illustrating the derivation of multivariable control equation: integral systems.

The plant equation is as before

$$D_i(p)Y_i(p)=K_i e^{-\tau_i p}\left[Y_i'(p)-\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)+f_i(p)\right]. \tag{4.36}$$

The equation of the measurement device

$$X_i'(p)pR_i(p)=\mu_i\{[Y_{i\,ref}(p)-Y_i(p)]+\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)[Y_{k\,ref}(p)-Y_k(p)]\}. \tag{4.37}$$

The amplifier equation

$$X_i''(p)=K_{i\,a}\left[X_i'(p)-\frac{F_{ni}(p)}{F_{mi}(p)}Y_i(p)\right]. \tag{4.38}$$

The controller equation

$$Q_i(p)Y_i'(p)=\delta_i X_i''(p). \tag{4.39}$$

Eliminating $Y_i'(p)$, $X_i'(p)$ and $X_i''(p)$ between (4.36)—(4.39), we obtain after simple manipulations the following equation for $Y_i(p)$:

$$\begin{aligned}
&[D_i(p)R_i(p)Q_i(p)F_{mi}(p)pe^{\tau_i p}+K_{i\,a}\delta_i R_i(p)Q_i(p)pF_{ni}(p)e^{\tau_i p}+\\
&+K_iK_{i\,a}\delta_i\mu_i F_{mi}(p)]Y_i(p)+\mu_i K_{i\,a}K_i\delta_i F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)Y_k(p)+\\
&+K_iR_i(p)Q_i(p)pF_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)+\\
&+\delta_i K_{i\,a}K_iR_i(p)pF_{ni}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=\\
&=K_iK_{i\,a}\delta_i\mu_i F_{mi}(p)Y_{i\,a}(p)+K_i\mu_i K_{i\,a}\delta_i F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)Y_{k\,ref}(p)+\\
&+K_i[R_i(p)pQ_i(p)F_{mi}(p)+R_i(p)pF_{ni}(p)\delta_i K_{i\,a}]f_i(p).
\end{aligned} \tag{4.40}$$

Putting $i = 1, 2, \ldots, n$, we obtain the complete set of equations which describe the dynamic properties of the multivariable structure under discussion. To reduce the set of equations to matrix form, we write

$$\left.\begin{aligned} &D_i(p)\,R_i(p)\,F_{mi}(p)\,pe^{\tau_i p} = a_i(p), \quad \delta_i K_{i\,a} = K_{i\,st}, \\ &R_i(p)\,Q_i(p)\,F_{ni}(p)\,pe^{\tau_i p} = b_i(p), \\ &R_i(p)\,F_{ni}(p)\,p = g_i(p), \quad K_i K_{i\,a}\delta_i\mu_i = K_{i\,tot}, \\ &K_i R_i(p)\,p\,Q_i(p)\,F_{mi}(p) = c_i(p), \\ &a_i(p) + K_{i\,st}\,b_i(p) + K_{i\,tot}F_{mi}(p) = a_{ii}(p). \end{aligned}\right\} \tag{4.41}$$

The equations can now be written in the form

$$\begin{aligned} &a_{ii}(p)\,Y_i(p) + K_{i\,tot}F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)\,Y_k(p) + \\ &+ c_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\,Y_k(p) + K_{i\,st}\,K_i g_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)\,Y_k(p) = \\ &= K_{i\,tot}F_{mi}(p)\,Y_{i\,ref}(p) + K_{i\,tot}F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}(p)\,Y_{k\,ref}(p) + \\ &+ [c_i(p) + K_i K_{i\,st}\,g_i(p)]\,f_i(p) \qquad (i = 1, 2, \ldots, n), \end{aligned} \tag{4.42}$$

or in matrix notation

$$AY = BY_{ref} + CF, \tag{4.43}$$

$$A = \begin{Vmatrix} a_{11}(p) & R_{12} & \ldots & R_{1n} \\ R_{21} & a_{22}(p) & \ldots & R_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ R_{n1} & & \ldots & a_{nn}(p) \end{Vmatrix}, \tag{4.44}$$

where

$$\begin{aligned} R_{12} &= K_{1\,tot}F_{m1}(p)\,r_{12} + (c_1(p) + K_1K_{1\,st}\,g_1(p))\,\alpha_{12}(p), \\ R_{1n} &= K_{1\,tot}F_{m1}(p)\,r_{1n} + (c_1(p) + K_1K_{1\,st}\,g_1(p))\,\alpha_{1n}(p), \\ R_{21} &= K_{2\,tot}F_{m2}(p)\,r_{21} + (c_2(p) + K_2K_{2\,st}\,g_2(p)\,\alpha_{21}(p), \\ R_{2n} &= K_{2\,tot}F_{m2}(p)\,r_{2n} + (c_2(p) + K_2K_{2\,st}\,g_2(p))\,\alpha_{2n}(p), \\ R_{n1} &= K_{n\,tot}F_{mn}(p)\,r_{n1}(p) + (c_n(p) + K_nK_{n\,st}\,g_n(p))\,\alpha_{n1}(p), \end{aligned}$$

$$B = \begin{Vmatrix} K_{1\,tot}F_{m1}(p) & K_{1\,tot}F_{m1}(p)\,r_{12}(p) & \ldots & K_{1\,tot}F_{m1}(p)\,r_{1n}(p) \\ K_{2\,tot}F_{m2}(p) & K_{2\,tot}F_{m2}(p) & \ldots & K_{2\,tot}F_{m2}(p)\,r_{2n}(p) \\ \cdots & \cdots & \cdots & \cdots \\ K_{n\,tot}F_{m2}(p)\,r_{n1} & \ldots & \ldots & K_{n\,tot}F_{mn}(p) \end{Vmatrix}, \tag{4.45}$$

$$C = \begin{Vmatrix} c_1(p) + K_1K_{1\,st}\,g_1(p) & 0 & \ldots & 0 \\ 0 & c_2(p) + K_2K_{2\,st}\,g_2(p) & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & c_n(p) + K_nK_{n\,st}\,g_n(p) \end{Vmatrix}, \tag{4.46}$$

$$Y = \begin{Vmatrix} Y_1(p) \\ Y_2(p) \\ \cdots \\ Y_n(p) \end{Vmatrix}, \quad Y_{ref} = \begin{Vmatrix} Y_{1\,ref}(p) \\ Y_{2\,ref}(p) \\ \cdots \\ Y_{n\,ref}(p) \end{Vmatrix}, \quad F = \begin{Vmatrix} f_1(p) \\ f_2(p) \\ \cdots \\ f_n(p) \end{Vmatrix}. \tag{4.47}$$

We will now derive an expression for the j-th controlled variable.

From (4.43) we have

$$Y = A^{-1}(BY_{\text{ref}} + CF). \tag{4.48}$$

Expanding (4.48) and proceeding along the same lines as in the previous case, we find

$$Y = \frac{1}{\Delta}\begin{Vmatrix} \sum_{\rho=1}^{n}(-1)^{1+\rho}A_{1\rho}K_{\rho\text{tot}}F_{m\rho}(p)\sum_{k=1}^{n}r_{\rho k}(p)\,Y_{k\,\text{ref}}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\rho=1}^{n}(-1)^{j+\rho}A_{j\rho}K_{\rho\text{tot}}F_{m\rho}(p)\sum_{k=1}^{n}r_{\rho k}(p)\,Y_{k\,\text{ref}}(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\rho=1}^{n}(-1)^{n+\rho}A_{n\rho}K_{\rho\text{tot}}F_{m\rho}(p)\sum_{k=1}^{n}r_{\rho k}(p)\,Y_{k\,\text{ref}}(p) \end{Vmatrix} + \\ + \frac{1}{\Delta}\begin{Vmatrix} \sum_{\rho=1}^{n}(-1)^{\rho+1}A_{1\rho}(c_\rho(p)+K_\rho K_{\rho\ st}\,b_\rho(p))f_\rho(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\rho=1}^{n}(-1)^{\rho+j}A_{j\rho}(c_\rho(p)+K_\rho K_{\rho\ st}\,b_\rho(p))f_\rho(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\rho=1}^{n}(-1)^{\rho+n}A_{n\rho}(c_\rho(p)+K_\rho K_{\rho\ st}\,b_\rho(p))f_\rho(p) \end{Vmatrix}. \tag{4.49}$$

Here all $r_{\rho\rho}=1$. From (4.49) we easily obtain an equation for any j-th controlled variable:

$$Y_j(p) = \frac{1}{\Delta}\left\{\sum_{\rho=1}^{n}(-1)^{\rho+j}A_{j\rho}\left[K_{\rho\text{tot}}F_{m\rho}\sum_{k=1}^{n}r_{\rho k}(p)\,Y_{k\,\text{ref}}(p) + [c_\rho(p) + \right.\right. \\ \left.\left. + K_\rho K_{\rho\ st}\,g_\rho(p)]f_\rho(p)\right]\right\}. \tag{4.50}$$

In particular, in a three-variable system, we have for the second controlled variable

$$Y_2(p) = \frac{1}{\Delta_3}\left\{\sum_{\rho=1}^{3}(-1)^{2+\rho}A_{2\rho}\left[K_{\rho\text{tot}}F_{m\rho}(p)\sum_{k=1}^{3}r_{\rho k}(p)\,Y_{\rho\,\text{ref}}(p) + \right.\right. \\ \left.\left. + (c_\rho(p) + K_\rho K_{\rho\ st}\,b_\rho(p))f_\rho(p)\right]\right\}. \tag{4.51}$$

We write equation (4.51) in expanded form:

$$\Delta_3 = \begin{vmatrix} a_{11}(p) & a_{12}(p) & a_{13}(p) \\ a_{21}(p) & a_{22}(p) & a_{23}(p) \\ a_{31}(p) & a_{32}(p) & a_{33}(p) \end{vmatrix},$$

$$A_t = \begin{Vmatrix} a_{11}(p) & a_{21}(p) & a_{31}(p) \\ a_{12}(p) & a_{22}(p) & a_{32}(p) \\ a_{13}(p) & a_{23}(p) & a_{33}(p) \end{Vmatrix}.$$

Here

$$a_{12}(p) = K_{1\text{tot}}F_{1m}(p)\,r_{12}(p) + [c_1(p) + K_1K_{1\ st}\,g_1(p)]\,\alpha_{12}(p),$$
$$a_{13}(p) = K_{1\text{tot}}F_{1m}(p)\,r_{13}(p) + [c_1(p) + K_1K_{1\ st}\,g_1(p)]\,\alpha_{13}(p),$$
$$a_{21}(p) = K_{2\text{tot}}F_{2m}(p)\,r_{21}(p) + [c_2(p) + K_2K_{2\ st}\,g_2(p)]\,\alpha_{21}(p),$$
$$a_{23}(p) = K_{2\text{tot}}F_{2m}(p)\,r_{23}(p) + [c_2(p) + K_2K_{2\ st}\,g_2(p)]\,\alpha_{23}(p),$$
$$a_{31}(p) = K_{3\text{tot}}F_{3m}(p)\,r_{31}(p) + [c_3(p) + K_3K_{3\ st}\,g_3(p)]\,\alpha_{31}(p),$$
$$a_{32}(p) = K_{3\text{tot}}F_{3m}(p)\,r_{32}(p) + [c_3(p) + K_3K_{3\ st}\,g_3(p)]\,\alpha_{32}(p),$$

whence

$$A_{21}=\begin{Vmatrix} R_{21} & R_{13} \\ R_{23} & a_{33}(p) \end{Vmatrix},\quad A_{23}=\begin{Vmatrix} a_{11}(p) & R'_{12} \\ R'_{21} & R'_{22} \end{Vmatrix},\quad A_{22}=\begin{Vmatrix} a_{11}(p) & R''_{12} \\ R''_{21} & a_{33}(p) \end{Vmatrix},$$

where

$$\begin{aligned}
R_{21} &= K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p)+(c_2(p)+K_2K_{2\ st}\,g_2(p))\,\alpha_{21}(p)\\
R_{31} &= K_{3\,\mathrm{tot}}F_{m3}(p)\,r_{31}(p)+(c_3(p)+K_3K_{3\ st}\,g_3(p))\,\alpha_{31}(p),\\
R_{23} &= K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{23}(p)+(c_2(p)+K_{2\ st}K_2g_2(p))\,\alpha_{23}(p),\\
R'_{12} &= K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p)+(c_2(p)+K_2K_{2\ st}\,g_2(p))\,\alpha_{21}(p),\\
R'_{21} &= K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{13}(p)+(c_1(p)+K_1K_{1\ st}\,g_1(p))\,\alpha_{13}(p),\\
R'_{22} &= K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{23}(p)+(c_2(p)+K_2K_{2\ st}\,g_2(p))\,\alpha_{23}(p),\\
R''_{12} &= K_{3\,\mathrm{tot}}F_{m3}(p)\,r_{31}(p)+(c_3(p)+K_3K_{3\ st}\,g_3(p))\,\alpha_{31}(p),\\
R''_{21} &= K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{13}(p)+(c_1(p)+K_1K_{1\ st}\,g_1(p))\,\alpha_{13}(p),
\end{aligned}$$

and

$$\begin{aligned}
Y_2(p)=\frac{1}{\Delta}\{&-([K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p)+(c_2(p)+\\
&+K_2K_{2\ st}\,g_2(p))\,\alpha_{21}(p)]\,a_{33}(p)-[K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{23}(p)+\\
&+(c_2(p)+K_2K_{2\ st}\,g_2(p))\,\alpha_{23}(p)]\,[K_{3\,\mathrm{tot}}F_{m3}(p)\,r_{31}(p)+\\
&+(c_3(p)+K_3K_{3\ st}\,g_3(p))\,\alpha_{31}(p)])\,(K_{1\,\mathrm{tot}}F_{m1}(p)\,Y_{1\,\mathrm{ref}}(p)+\\
&+K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{12}Y_{2\,\mathrm{ref}}(p)+K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{13}(p)\,Y_{3\,\mathrm{ref}}(p)+\\
&+(c_1(p)+K_1K_{1\ st}\,b_1(p)\,f_1(p))+(a_{11}(p)\,a_{33}(p)-\\
&-[K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{13}(p)+(c_1(p)+K_{1\ st}\,g_1(p))\,\alpha_{13}(p)]\times\\
&\times[K_{3\,\mathrm{tot}}F_{m3}(p)\,r_{31}(p)+(c_3(p)+K_3K_{3\ st}\,g_3(p))\,\alpha_{31}(p)])-\\
&-(K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p)\,Y_{1\,\mathrm{ref}}(p)+K_{2\,\mathrm{tot}}F_{m2}(p)\,Y_{2\,\mathrm{ref}}(p)+\\
&+K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{23}(p)\,Y_{3\,\mathrm{ref}}(p)+(c_2(p)+K_{2\ st}K_2g_2(p))\,f_2(p))-\\
&-(a_{11}(p)\,[K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{23}(p)+(c_2(p)+\\
&+K_{2\ st}K_2g_2(p))\,\alpha_{23}(p)]-[K_{1\,\mathrm{tot}}F_{m1}(p)\,r_{13}(p)+\\
&+(c_1(p)+K_{1\ st}K_1g_1(p))\,\alpha_{13}(p)]\,[K_{2\,\mathrm{tot}}F_{m2}(p)\,r_{21}(p)+\\
&+(c_2(p)+K_2K_{2\ st}\,g_2(p))\,\alpha_{21}(p)])\times\\
&\times(K_{3\,\mathrm{tot}}F_{m3}(p)\,[r_{31}(p)\,Y_{1\,\mathrm{ref}}(p)+r_{32}(p)\,Y_{2\,\mathrm{ref}}(p)+\\
&+Y_{3\,\mathrm{ref}}(p)]+(c_3(p)+K_3K_{3\ st}\,g_3(p)\,f_3(p))\}.
\end{aligned}\tag{4.52}$$

The steady-state value of the i-th controlled variable can be readily found from (4.52). First, however, we should determine $A_{ji}(0)$ from the transpose making use of standard rules of matrix algebra.

As an example, we calculate the steady-state value in a three-variable system. From (4.52) we find

$$\begin{aligned}
Y_2(0)=\frac{1}{\Delta}\{&(-K_{2\,\mathrm{tot}}r_{21}K_{3\,\mathrm{tot}}+K_{2\,\mathrm{tot}}r_{23}K_{3\,\mathrm{tot}}r_{31})\,(K_{1\,\mathrm{tot}}Y_{1\,\mathrm{ref}}+\\
&+K_{1\,\mathrm{tot}}r_{12}Y_{2\,\mathrm{ref}}+K_{1\,\mathrm{tot}}r_{13}Y_{3\,\mathrm{ref}})+(K_{1\,\mathrm{tot}}K_{3\,\mathrm{tot}}-K_{1\,\mathrm{tot}}r_{13}K_{3\,\mathrm{tot}}r_{31})\times\\
&\times(K_{2\,\mathrm{tot}}r_{21}Y_{1\,\mathrm{ref}}+K_{2\,\mathrm{tot}}Y_{2\,\mathrm{ref}}+K_{2\,\mathrm{tot}}r_{23}Y_{3\,\mathrm{ref}})-\\
&-(K_{1\,\mathrm{tot}}K_{2\,\mathrm{tot}}r_{23}-K_{1\,\mathrm{tot}}r_{13}K_{2\,\mathrm{tot}}r_{21})\times\\
&\times(K_{3\,\mathrm{tot}}r_{31}Y_{1\,\mathrm{ref}}+K_{3\,\mathrm{tot}}r_{32}Y_{2\,\mathrm{ref}}+K_{3\,\mathrm{tot}}Y_{3\,\mathrm{ref}})\},
\end{aligned}\tag{4.53}$$

where

$$\Delta=\begin{vmatrix} K_{1\,\mathrm{tot}} & K_{1\,\mathrm{tot}}r_{12} & K_{1\,\mathrm{tot}}r_{13} \\ K_{2\,\mathrm{tot}}r_{21} & K_{2\,\mathrm{tot}} & K_{2\,\mathrm{tot}}r_{23} \\ K_{3\,\mathrm{tot}}r_{31} & K_{2\,\mathrm{tot}}r_{32} & K_{3\,\mathrm{tot}} \end{vmatrix}.\tag{4.54}$$

Thus, in addition to the general steady-state properties of the system, we have derived working relations for the determination of the controlled variables.

§ 4.2. SYSTEM DYNAMICS

We now proceed with a discussion of the dynamic properties of proportional and integral configurations.

(a) PROPORTIONAL SYSTEMS

The system depicted in Figure 4.1 will retain its stability when the gain parameters of the elements in the stabilizer loop are increased indefinitely. This structural property is expressed mathematically in the form (see (4.2))

$$n_i - m_i + r_i + q_i \leqslant 2.$$

Having chosen a stabilizer, we make connections that satisfy the structural criterion above and thus create a system which in principle remains stable despite an indefinite increase in the subsystem gains. To ensure realizability, the degenerate and the auxiliary equation should of course satisfy the stability criteria. Since the structural stability requirement is a priori satisfied, we have to choose the stabilizer parameters and the gains of the starting single-loop system so that all the coefficients of the degenerate and the auxiliary equation meet the respective stability criteria. It is at this stage that we should take steps to ensure not only the stability but also the desired dynamic characteristics (speed and transients) of the control system.

Let us consider the fundamental proportional-control structures. It will be assumed throughout that the stabilizer uses passive elements only. For this reason the degree of p in the numerator of the stabilizer rational-fractional function is equal to or less than the degree of p in the denominator. In our nomenclature, we may thus write

$$n_i \leqslant m_i. \tag{4.55}$$

From (4.2) and (4.55) it follows that the degree of the self-operator of the stabilized section must not exceed 2.

We now return to equation (4.7) which describes the j-th controlled variable of the structure shown in Figure 4.1. Let each subsystem be made up of aperiodic and amplifying elements. Condition (4.2) is satisfied in the following two cases: either the self-operators have the form

$$R_i(p)\,Q_i(p) = (1 + T_{i_1} p)(1 + T_{i_2} p),$$

or, if one of the time constants is zero, we have a general self-operator of the form

$$R_i(p)\,Q_i(p) = (1 + T_i p).$$

In the former case

$$n_1 - m_1 = 0,$$

and in the latter it is permissible that

$$n_1 - m_1 = 1.$$

Let us consider the first of the two cases. Equation (4.7) takes the form

$$\begin{aligned}
&\{D_i(p)[(1+T_{i\,r}p)(1+T_{i\,q}p)e^{\tau_i p}F_{mi}(p)+K_{i\ st}F_{ni}(p)e^{\tau_i p}]+\\
&+K_{i\ st}K_{i\ deg}F_{mi}(p)\}Y_i(p)+K_{i\,tot}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_k(p)+\\
&+(1+T_{i\,q}p)(1+T_{i\,r}p)F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)+\\
&+K_{i\ st}F_{ni}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=\\
&=K_{i\,tot}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_{k\,ref}(p)+K_{i\,tot}F_{im}(p)Y_{i\,ref}(p)+\\
&+(1+T_{i\,r}p)(1+T_{i\,q}p)F_{im}(p)f_i(p)+K_{i\ st}F_{in}(p)f_i(p). \qquad (4.56)
\end{aligned}$$

Dividing (4.56) by $K_{i\ st}$ and assuming a sufficiently large $K_{i\ st}$, we put $\frac{1}{K_{i\ st}}=m_i$ and write

$$\begin{aligned}
&\{D_i(p)[m_i(1+T_{i\,r}p)(1+T_{i\,q}p)F_{im}(p)e^{\tau_i p}+F_{in}(p)e^{\tau_i p}]+\\
&+K_{i\,deg}F_{im}(p)\}Y_i(p)+K_{i\ deg}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_k(p)+\\
&+m_i(1+T_{i\,r}p)(1+T_{i\,q}p)F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)+\\
&+F_{in}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=K_{i\ deg}F_{im}(p)Y_{i\,ref}(p)+\\
&+K_{i\ deg}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_{k\,ref}(p)+\\
&+m(1+T_{i\,r}p)(1+T_{i\,q}p)F_{im}(p)f_i(p)+F_{in}(p)f_i(p). \qquad (4.57)
\end{aligned}$$

We can now find the matrix equation of the output vector as a function of the small parameter m. Putting

$$\left.\begin{aligned}
m(1+T_{i\,r}p)(1+T_{i\,q}p)F_{mi}(p)D_i(p)e^{\tau_i p}&=ma_i(p),\\
D_i(p)F_{in}(p)e^{\tau_i p}&=b_i(p),\\
m(1+T_{i\,q}p)(1+T_{i\,r}p)F_{im}(p)&=mc_i(p),
\end{aligned}\right\} \qquad (4.58)$$

we rewrite (4.57) in the form

$$\begin{aligned}
&[ma_i(p)+b_i(p)+K_{i\ deg}F_{im}(p)]Y_i(p)+\\
&+K_{i\ deg}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_k(p)+mc_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)+\\
&+F_{in}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=K_{i\ deg}F_{im}(p)Y_{i\,ref}(p)+\\
&+K_{i\ deg}F_{im}(p)\sum_{\substack{k=1\\k\neq i}}^{n}r_{ik}(p)Y_{k\,ref}(p)+\\
&+mc_i(p)f_i(p)+F_{in}(p)f_i(p) \qquad (i=1,\ 2,\ \ldots,\ n). \qquad (4.59)
\end{aligned}$$

The matrix form of equation (4.59) is

$$AY = K_{\deg} F_m Y_{\text{ref}} + BY_{\text{ref}} + NF. \tag{4.60}$$

Here

$$A = \begin{Vmatrix} ma_1(p) + b_1(p) + & \dots & K_{1\deg} F_{1m}(p) r_{1n} + \\ + K_{1\deg} F_{1m}(p) & & + [mc_1(p) + F_{1n}(p)] \alpha_{1n} \\ \dots & & \dots \\ K_{j\deg} F_{jm}(p) r_{j1} + & \dots & K_{j\deg} F_{jm}(p) r_{jn} + \\ + [mc_j(p) + F_{jn}(p) \alpha_{j1}(p)] & & + [mc_j(p) + F_{jn}(p)] \alpha_{jn} \\ \dots & & \dots \\ K_{n\deg} F_{nm}(p) r_{n1} + & \dots & ma_n(p) + b_n(p) + \\ + [mc_n(p) + F_{nn}(p)] \alpha_{n1} & & + K_{n\deg} F_{nm}(p) \end{Vmatrix}, \tag{4.61}$$

$$B = \begin{Vmatrix} 0 & K_{1\deg}(p) F_{1m}(p) r_{12}(p) & \dots & K_{1\deg} F_{1m}(p) r_{1n}(p) \\ K_{2\deg} F_{2m}(p) r_{21}(p) & 0 & \dots & K_{2\deg} F_{2m}(p) r_{2n}(p) \\ \dots & \dots & \dots & \dots \\ K_{\deg} F_{nm}(p) r_{n1}(p) & \dots & \dots & 0 \end{Vmatrix}, \tag{4.62}$$

$$N = \begin{Vmatrix} mc_1(p) + F_{1n}(p) & 0 & \dots & 0 \\ 0 & mc_2(p) + F_{2n}(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & mc_n(p) + F_{nn}(p) \end{Vmatrix}. \tag{4.63}$$

From (4.60),

$$Y = A^{-1}[K_{\deg} F_m Y_{\text{ref}} + BY_{\text{ref}} + NF]. \tag{4.64}$$

Consider the degenerate vector equation. Since condition (4.2) is satisfied, we assume that the auxiliary equation of second kind meets the stability requirements and thus obtain the degenerate case from (4.64) putting $m = 0$. From (4.64), (4.63), (4.62), and (4.61) we have

$$Y_{\deg} = A_{\deg}^{-1}[K_{\deg} F_m Y_{\text{ref}} + BY_{\text{ref}} + N_{\deg} F], \tag{4.65}$$

where

$$A_{\deg} = \begin{Vmatrix} b_1 + K_{1\deg} F_{1m} & K_{1\deg} F_{1m} r_{12} + F_{1n} \alpha_{12} \dots K_{1\deg} F_{1m} r_{1n} + F_{1n} \alpha_{1n} \\ \dots & \dots \\ K_{n\deg} F_{nm} r_{n1} + F_{nn} \alpha_{n1} & \dots \quad \dots b_n + K_{n\deg} F_{nm} \end{Vmatrix}, \tag{4.66}$$

$$N_{\deg} = \begin{Vmatrix} F_{1n} & 0 & \dots & 0 \\ 0 & F_{2n} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & F_{nn} \end{Vmatrix}. \tag{4.67}$$

The transpose of (4.66) is

$$A_{\deg t} = \begin{Vmatrix} b_1 + K_{1\deg} F_{1m} & K_{2\deg} F_{2m} r_{21} + F_{2n} \alpha_{21} \dots K_{n\deg} F_{nm} r_{n1} + F_{nn} \alpha_{n1} \\ \dots & \dots \\ K_{1\deg} F_{1m} r_{1n} + F_{1n} \alpha_{1n} & \dots \quad \dots b_n + K_{n\deg} F_{nn} \end{Vmatrix}. \tag{4.68}$$

The elements of $A_{\deg}^{-1}$ are found from (4.68). Inserting the respective expressions for the matrices in (4.65) and multiplying, we obtain

$$Y_{\deg} = \frac{1}{\Delta} \begin{Vmatrix} A_{11\deg} & -A_{12\deg} & \dots & (-1)^{1+n} A_{1n\deg} \\ \dots & \dots & \dots & \dots \\ (-1)^{j+1} A_{j1\deg} & (-1)^{j+2} A_{j2\deg} & \dots & (-1)^{j+n} A_{jn\deg} \\ \dots & \dots & \dots & \dots \\ (-1)^{1+n} A_{n1\deg} & \dots & \dots & A_{nn\deg} \end{Vmatrix} \times$$

$$\times \left\| \begin{array}{l} K_{1b}F_{1m}Y_{1\,\mathrm{ref}} + K_{1\,\deg}F_{1m}r_{12}Y_{2\,\mathrm{ref}} + \ldots + K_{1\,\deg}F_{1m}r_{1n}Y_{n\,\mathrm{ref}} + F_{1n}f_1 \\ K_{2\,\deg}F_{2m}r_{21}Y_{1\,\mathrm{ref}} + K_{2\,\deg}F_{2m}Y_{2\,\mathrm{ref}} + \ldots + K_{2\,\deg}F_{2m}r_{2n}Y_{n\,\mathrm{ref}} + F_{2n}f_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K_{n\,\deg}F_{nm}r_{n1}Y_{1\,\mathrm{ref}} + \ldots + K_{n\,\deg}F_{nm}Y_{n\,\mathrm{ref}} + F_{nn}f_n \end{array} \right\| =$$

$$= \frac{1}{\Delta} \left\| \begin{array}{l} A_{11\,\deg}\left(K_{1\,\deg}F_{1m}\sum_{i=1}^{n} r_{1i}Y_{i\,\mathrm{ref}} + F_{1n}f_1\right) + \ldots \\ \qquad \ldots + (-1)^{1+n}A_{1n\,\deg}\left(K_{n\,\deg}F_{nm}\sum_{i=1}^{n} r_{ni}Y_{i\,\mathrm{ref}} + F_{nn}f_n\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-1)^{j+1}A_{j1\,\deg}\left(K_{1\,\deg}F_{1m}\sum_{i=1}^{n} r_{1i}Y_{i\,\mathrm{ref}} + F_{1n}f_1\right) + \ldots \\ \qquad \ldots + (-1)^{1+n}A_{jn\,\deg}\left(K_{n\,\deg}F_{nm}\sum_{i=1}^{n} r_{ni}Y_{i\,\mathrm{ref}} + F_{nn}f_n\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (-1)^{n+1}A_{n1\,\deg}\left(K_{1\,\deg}F_{1m}\sum r_{1i}Y_{i\,\mathrm{ref}} + F_{1n}f_1\right) + \ldots \\ \qquad \ldots + A_nK_{n\,\deg}F_{nm}\left(\sum_{i=1}^{n} r_{ni}Y_{i\,\mathrm{ref}} + F_{nn}f_n\right) \end{array} \right\|$$

(everywhere $r_{ii} = 1$),

whence follows the degenerate equation for the j-th controlled variable:

$$Y_j = \frac{1}{\Delta}\sum_{i=1}^{n}(-1)^{j+i}A_{ji\,\deg}\left[K_{i\,\deg}F_{im}\sum_{\rho} r_{i\rho}Y_{\rho\mathrm{ref}} + F_{in}f_i\right]. \qquad (4.69)$$

Comparison of (4.69), (4.50), and (4.18) shows that these equations are identical. They all have the same structure, differing only in the numerical values of Δ, A_{ji}, and other coefficients. The expression for the full value of Y_j (nondegenerate) is obviously of the same form as (4.69), with the difference that its components include coefficients that depend on m. Any of the controlled variables can be found from (4.69).

Let us consider the properties of the degenerate equation for the first controlled variable in a plant- and transducer-coupled three-variable system. From (4.69) for $n = 3$ we have

$$\begin{aligned} Y_1 = \frac{1}{\Delta_3}[&A_{11\,\deg}[K_{1\,\deg}F_{1m}(Y_{1\,\mathrm{ref}} + r_{12}Y_{2\,\mathrm{ref}} + r_{13}Y_{3\,\mathrm{ref}}) + F_{1n}f_1] - \\ &- A_{12\,\deg}[K_{2\,\deg}F_{2m}(r_{21}Y_{1\,\mathrm{ref}} + Y_{2\,\mathrm{ref}} + r_{23}Y_{3\,\mathrm{ref}}) + F_{2n}f_2] + \\ &+ A_{13\,\deg}[K_{3\,\deg}F_{3m}(r_{31}Y_{1\,\mathrm{ref}} + r_{32}Y_{2\,\mathrm{ref}} + Y_{3\,\mathrm{ref}}) + F_{3n}f_3]]. \end{aligned} \qquad (4.70)$$

Writing out the expressions entering (4.70), we have

$$\Delta_3 = \left| \begin{array}{lll} b_1 + K_{1\,\deg}F_{1m} & K_{1\,\deg}F_{1m}r_{12} + F_{1n}\alpha_{12} & K_{1\,\deg}F_{1m}r_{13} + F_{1n}\alpha_{13} \\ K_{2\,\deg}F_{2m}r_{21} + F_{2n}\alpha_{21} & b_2 + K_{2\,\deg}F_{2m} & K_{2\,\deg}F_{2m}r_{23} + F_{2n}\alpha_{23} \\ K_{3\,\deg}F_{3m}r_{31} + F_{3n}\alpha_{31} & K_{3\,\deg}F_{3m}r_{32} + F_{3n}\alpha_{32} & b_3 + K_{3\,\deg}F_{3m} \end{array} \right|, \qquad (4.71)$$

$$\left.\begin{array}{l} A_{11} = \left| \begin{array}{ll} b_2 + K_{2\,\deg}F_{2m} & K_{3\,\deg}F_{3m}r_{32} + F_{3n}\alpha_{32} \\ K_{2\,\deg}F_{2m}r_{23} + F_{2n}\alpha_{23} & b_3 + K_{3\,\deg}F_{3m} \end{array} \right|, \\ A_{12} = \left| \begin{array}{ll} K_{1\,\deg}F_{1m}r_{12} + F_{1n}\alpha_{12} & K_{3\,\deg}F_{3m}r_{32} + F_{3n}\alpha_{32} \\ K_{1\,\deg}F_{1m}r_{13} + F_{1n}\alpha_{13} & b_3 + K_{3\,\deg}F_{3n} \end{array} \right|, \\ A_{13} = \left| \begin{array}{ll} K_{1\,\deg}F_{1m}r_{12} + F_{1n}\alpha_{12} & b_2 + K_{2\,\deg}F_{2m} \\ K_{1\,\deg}F_{1m}r_{13} + F_{1n}\alpha_{13} & K_{2\,\deg}F_{2m}r_{23} + F_{2n}\alpha_{23} \end{array} \right|. \end{array}\right\} \qquad (4.72)$$

The first structural conclusion which obtains from the general equation for the j-th variable as determined via the degenerate equation (4.69), and which is likewise applicable to any particular case of a degenerate system, is the following: a degenerate system of the given structure is characterized by a multicoupled dynamics.

The coupling between the controlled variables in this case is determined by the properties of the controlled plant (the coefficients α_{ih}) and the additional interconnections artificially introduced into the system (the coefficients r_{ih}). As regards transducer coupling, it is artificial and is thus specified by the particular features of the technical problem at hand; the contribution from this coupling to system dynamics should thus be elucidated for each individual case separately. Note that transducer coupling is introduced to ensure a certain resultant variation of all the controlled coordinates as a function of variation of each individual coordinate. This interrelationship ensues primarily from the fact that a change in any controlled variable modifies the setting for all the other controlled variables. However, as is clear from the expressions for Δ, system stability requirements should be kept in mind in choosing r_{ih}, since the characteristic equation $\Delta = 0$ depends on r_{ih}.

Let us consider three different cases for $n = 3$, specifically (1) $r_{ih} = 0$, $\alpha_{ih} \neq 0$, (2) $r_{ih} \neq 0$, $\alpha_{ih} = 0$, and (3) $r_{ih} \neq 0$, $\alpha_{ih} \neq 0$.

Case 1. $r_{ih} = 0$, $\alpha_{ih} \neq 0$.

This case corresponds to an ordinary plant-coupled multivariable system. From (4.70) we have

$$Y_1 = \frac{1}{\Delta_3'} A_{11}' (K_{1\,\mathrm{deg}} F_{1m} Y_{1\,\mathrm{ref}} + F_{1n} f_1) - A_{12}' (K_{2\,\mathrm{deg}} F_{2m} Y_{2\,\mathrm{ref}} + \\ + F_{2n} f_2) + A_{13}' (K_{3\,\mathrm{deg}} F_{3m} Y_{3\,\mathrm{ref}} + F_{3n} f_3), \tag{4.73}$$

where

$$\Delta_3' = \begin{vmatrix} b_1 + K_{1\,\mathrm{deg}} F_{1m} & F_{1n}\alpha_{12} & F_{1n}\alpha_{13} \\ F_{2n}\alpha_{21} & b_2 + K_{2\,\mathrm{deg}} F_{2m} & F_{2n}\alpha_{23} \\ F_{3n}\alpha_{31} & F_{3n}\alpha_{32} & b_3 + K_{3\,\mathrm{deg}} F_{3m} \end{vmatrix}, \tag{4.74}$$

$$A_{11}' = \begin{vmatrix} b_2 + K_{2\,\mathrm{deg}} F_{2m} & F_{3n}\alpha_{31} \\ F_{2n}\alpha_{23} & b_3 + K_{3\,\mathrm{deg}} F_{3m} \end{vmatrix}, \tag{4.75}$$

$$A_{12}' = \begin{vmatrix} F_{1n}\alpha_{12} & F_{3n}\alpha_{32} \\ F_{3n}\alpha_{31} & b_3 + K_{3\,\mathrm{deg}} F_{m3} \end{vmatrix}, \tag{4.75'}$$

and

$$A_{13}' = \begin{vmatrix} F_{1n}\alpha_{12} & b_2 + K_{2\,\mathrm{deg}} F_{2m} \\ F_{1n}\alpha_{13} & F_{2n}\alpha_{23} \end{vmatrix}. \tag{4.75''}$$

The closed-loop transfer function (not generalized, so that $f_i = 0$) is

$$K(p) = \frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{A_{11}'(p) K_{1\,\mathrm{deg}} F_{1m}(p) - A_{12}'(p) K_{2\,\mathrm{deg}} F_{2m}(p) \dfrac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)} +}{\Delta_3'(p)} \\ + \frac{A_{13}'(p) K_{3\,\mathrm{deg}} F_{3m}(p) \dfrac{Y_{3\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}}{\Delta_3'(p)}. \tag{4.76}$$

Let us find the expression of the D-decomposition curve for the gain factor $K_{1\,\mathrm{deg}}$. The determinant Δ is not affected by interchanging its rows and columns. This transposition will simplify further manipulations, and we therefore write the determinant Δ' in the form

$$\Delta_3' = \begin{vmatrix} b_1 + K_{1\,\mathrm{deg}} F_{1m} & F_{2n}\alpha_{21} & F_{3n}\alpha_{31} \\ F_{1n}\alpha_{12} & b_2 + K_{2\,\mathrm{deg}} F_{2m} & F_{3n}\alpha_{32} \\ F_{1n}\alpha_{13} & F_{2n}\alpha_{23} & b_3 + K_{3\,\mathrm{deg}} F_{3m} \end{vmatrix}. \qquad (4.77)$$

Expanding (4.77) in elements of the first row and making use of (4.75), we obtain for the characteristic equation of the system

$$[b_1(p) + K_{1\,\mathrm{deg}} F_{1m}(p)] A_{11}'(p) - F_{2n}(p)\alpha_{21} A_{12}'(p) + F_{3n}(p)\alpha_{31} A_{13}'(p)] = 0,$$

whence follows an equation of the D-decomposition curve for the gain $K_{1\,\mathrm{deg}}$ of the first-loop degenerate equation:

$$\bar{K}_{1\,\mathrm{deg}}(p) = -\frac{A_{11}'(p) b_1(p) + F_{3n}(p)\alpha_{31} A_{13}'(p) - F_{2n}(p)\alpha_{21} A_{12}'(p)}{F_{1m}(p) A_{11}'(p)} \quad (p = j\omega). \qquad (4.78)$$

Dividing the numerator and the denominator of (4.76) by $F_{1m}(p) A_{11}'(p)$, we find

$$\left.\begin{aligned} K_3(p) &= \frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{K_{1\,\mathrm{deg}} + N(p) + M(p)}{\varphi(p)}, \\ \text{where} \\ N(p) &= \frac{K_{2\,\mathrm{deg}} A_{12}'(p) \dfrac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)} F_{2m}(p)}{A_{11}'(p) F_{1m}(p)}, \\ M(p) &= K_{3\,\mathrm{deg}} \frac{F_{3m}(p) A_{13}'(p) \dfrac{Y_{3\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}}{F_{1m}(p) A_{11}'(p)}, \\ \varphi(p) &= K_{1\,\mathrm{deg}} + \frac{A_{11}'(p) b_1(p) - F_{2n}(p)\alpha_{21} A_{12}'(p) + F_{3n}(p)\alpha_{31} A_{13}'(p)}{F_{1m}(p) A_{11}(p)}. \end{aligned}\right\} \qquad (4.79)$$

Equation (4.79) fully specifies the dynamic properties of a three-variable degenerate system. A similar expression can be obtained for a degenerate system of n plant-coupled variables. By analogy with (4.79), we have for the n-variable case

$$K_l(p) = \frac{Y_l(p)}{Y_{l\,\mathrm{ref}}(p)} = \frac{K_{l\,\mathrm{deg}} + \sum\limits_{\substack{k=1 \\ k \neq l}}^{n} (-1)^{k+l} A_{lk}'(p) \dfrac{F_{km}(p)}{F_{lm}(p)} \cdot \dfrac{Y_{k\,\mathrm{ref}}(p)}{Y_{l\,\mathrm{ref}}(p)}}{K_{1\,\mathrm{deg}} + \bar{K}_{1\,\mathrm{deg}}}, \qquad (4.80)$$

where

$$\bar{K}_{1\,\mathrm{deg}} = -\frac{b_l(p) A_{ll}'(p) + \sum\limits_{\substack{k=1 \\ k \neq l}}^{n} (-1)^{l+k} F_{kn}\alpha_{ki}(p) A_{lk}'(p)}{F_{lm}(p) A_{ll}'(p)}. \qquad (4.81)$$

Equation (4.81) is the equation of the D-decomposition curve for the gain factor $K_{1\,\mathrm{deg}}$ of a degenerate n-variable system.

In (4.80) and (4.81), $A'_{ij}(p)$ is found from a determinant of degree n as previously in the particular case of a third-degree equation.

Let us consider expression (4.79) in more detail. For uncoupled controlled variables,

$$\alpha_{ik}=0$$

and it follows from (4.75) that

$$A'_{12}=A'_{13}=0.$$

Equation (4.79) thus takes the form

$$K_3(p)=\frac{K_{1\,\mathrm{deg}}}{K_{1\,\mathrm{deg}}+\dfrac{b_1(p)}{F_{1m}(p)}}. \tag{4.82}$$

The denominator in (4.82) is a sum of $K_{1\,\mathrm{deg}}$ plus the equation of the D-decomposition curve for $K_{1\,\mathrm{deg}}$. The numerator is $K_{1\,\mathrm{deg}}$ alone. We thus see that the D-decomposition curve fully describes the dynamic properties of the system in this case /39/.

Comparison of equations (4.79) and (4.82) shows that plant coupling always has a substantial influence on the dynamics of each subsystem. In the general case, the effect introduced by coupling may be advantageous (if coupling improves the dynamic properties of the given subsystem) or disadvantageous (when the dynamic properties deteriorate due to coupling).

From the general equation of the transfer function of an n-variable system (equation (4.80)) we see that the dynamics of the i-th subsystem cannot be determined from the D-decomposition curve alone. The D-decomposition curve should be supplemented in general by an auxiliary curve, the system dynamics being obtained from these two curves jointly. As an example, we shall calculate the fundamental dynamic properties of a two-variable system.

From (4.80) we have for the first controlled variable ($n=2$)

$$K_{1\,\mathrm{deg}}(p)=\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)}=\frac{K_{1\,\mathrm{deg}}-\dfrac{F_{2m}(p)}{F_{1m}(p)}\dfrac{A_{12}(p)}{A'_{11}(p)}\dfrac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}}{K_{1\,\mathrm{deg}}+\dfrac{b_1(p)\,A'_{11}(p)-F_{2n}(p)\,A'_{12}(p)\,\alpha_{21}}{F_{1m}(p)\,A'_{11}(p)}}. \tag{4.83}$$

Here

$$\left.\begin{aligned} A'_{11}&=b_2+K_{2\,\mathrm{deg}}F_{2m}=D_2(p)\,F_{2n}(p)+K_{2\,\mathrm{deg}}F_{2m}(p),\\ -A'_{12}&=-F_{2n}(p)\,\alpha_{21}. \end{aligned}\right\} \tag{4.84}$$

Substituting (4.84) in (4.83) we find

$$K_{1\,\mathrm{deg}}(p)=\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)}=\frac{K_{1\,\mathrm{deg}}-\dfrac{F_{2m}(p)}{F_{1m}(p)}\dfrac{F_{1n}\alpha_{12}}{N(p)}\cdot\dfrac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}}{K_{1\,\mathrm{deg}}+\dfrac{D_1(p)\,F_{1n}(p)\,N(p)}{F_{1m}(p)\,N(p)}-\dfrac{F_{2n}(p)\,\alpha_{21}F_{1n}(p)\,\alpha_{12}}{F_{1m}(p)\,N(p)}}, \tag{4.85}$$

where

$$N(p) = D_2(p)F_{2n}(p) + K_{2\,\text{deg}}F_{2m}(p).$$

We first construct the D-decomposition curve for $K_{1\,\text{deg}}$ assuming uncoupled variables. Thus,

$$\overline{K}_{1\,\text{deg}} = -\frac{D_1(p)F_{n1}(p)}{F_{1m}(p)}, \qquad p = j\omega. \tag{4.86}$$

We have previously assumed that the stabilized section is structurally representable as one or two aperiodic elements in series. We thus choose the following transfer function for the stabilizer:

$$\frac{F_{1n}(p)}{F_{1m}(p)} = \frac{\tau_1 p}{1+\tau_1 p}. \tag{4.87}$$

The plant transfer functions for the first and second controlled variables are chosen from

$$\left.\begin{aligned} D_1(p) &= a_0' p^3 + a_1' p^2 + a_2' p + a_3', \\ D_2(p) &= a_0'' p^3 + a_1'' p^2 + a_2'' p + a_3'' \end{aligned}\right\} \tag{4.88}$$

and

$$\frac{F_{2n}(p)}{F_{2m}(p)} = \frac{\tau_2 p}{1+\tau_2 p}.$$

We adopt the following (arbitrary) numerical values of the coefficients:

$$\tau_1 = 0.3 \text{ sec}, \quad \tau_2 = 0.2 \text{ sec},$$
$$a_0' = 0.001, \quad a_1' = 0.1, \quad a_2' = 1, \quad a_3' = 1,$$
$$a_0'' = 0.0001, \quad a_1'' = 0.001, \quad a_2'' = 0.1, \quad a_3'' = 0.1,$$
$$K_{2\,\text{deg}} = 5, \quad \alpha_{21} = 0.5, \quad Y_{1\,\text{ref}}(p) = Y_{2\,\text{ref}}(p).$$

Figure 4.3 (curve a) is the D-decomposition curve in the $K_{1\,\text{deg}}$ plane plotted from equation (4.86). As is shown in § 3.7, the system dynamics in this case can be obtained directly from the D-decomposition curve.

We now plot the D-decomposition curve and the auxiliary curve making use of (4.85). The equation of the D-decomposition curve in this case is

$$\overline{K}_{1\,\text{deg}} = \frac{D_1(p)F_{1n}(p)[D_2(p)F_{2n}(p) + K_{2\,\text{deg}}F_{2m}(p)]}{F_{1m}(p)[D_2(p)F_{n2}(p) + K_{2\,\text{deg}}F_{2m}(p)]} \quad (p = j\omega). \tag{4.89}$$

The equation of the auxiliary curve is

$$W_{1x}(p) = \frac{K_{1\,\text{deg}} - \dfrac{F_{2m}(p)}{F_{1m}(p)}F_{1n}(p)\alpha_{12}}{D_2(p)F_{n2}(p) + K_{2\,\text{deg}}F_{2m}(p)}. \tag{4.90}$$

Curve b in Figure 4.3 is the D-decomposition curve constructed from (4.89). The auxiliary curve in our particular case has virtually no effect on the system dynamics. Indeed, since $\frac{F_{2m}(p)}{F_{1m}(p)}$ is close to unity in the entire relevant frequency range, the numerator of (4.90) is close to $K_{1\,\text{deg}}$.

The D-decomposition curve thus provides information not only on system stability but also on the fundamental dynamic characteristics.

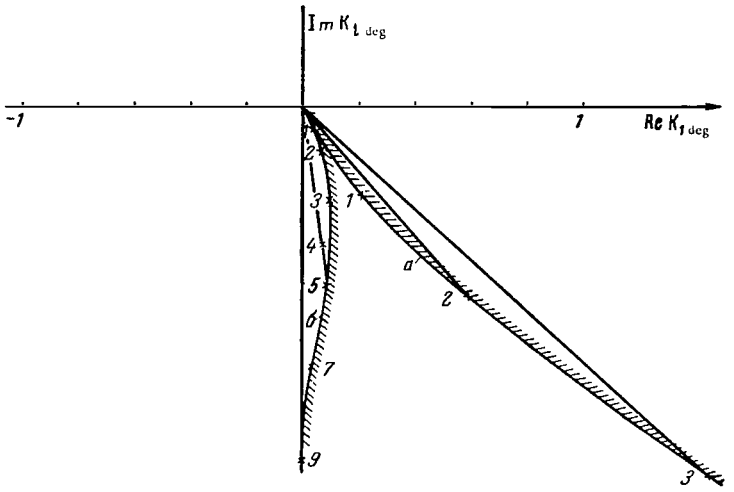

FIGURE 4.3. D-decomposition curve.

The D-decomposition curves suggest the following conclusions.

1. The region of stability of an isolated system is less than that of a coupled system (the intersection points of curves a and b in Figure 4.3 are not shown).

2. The positive-response bandwidth for the given K_1 value in a coupled system is substantially greater than that of an isolated system, whence it follows that the dynamic properties of a coupled system are substantially better than those of an isolated system.

3. It is clear from the preceding that in the case at hand the system should not be made noninteracting. This conclusion, however, is by no means applicable to other numerical values of the parameters.

The D-decomposition curves are tabulated numerically in Tables 4.1 and 4.2 for the two cases being considered. We see that $K_{2\,\mathrm{deg}}$ must not be ignored. In constructing the D-decomposition curve for $K_{1\mathrm{deg}}$, we put $K_{2\mathrm{deg}} = 5$. A change in this parameter substantially modifies the trend of the curve (see Tables 4.1 and 4.2). The D-decomposition curves should therefore be constructed for all $K_{i\mathrm{deg}}$, the appropriate value of $K_{i\,\mathrm{deg}}$ being picked out in accordance with the problem at hand.

The choice of the parameters may also substantially influence the auxiliary curve, as is clearly evident from Table 4.3. We see that in our case the auxiliary curve can be reduced to a single point, $K_{1\,\mathrm{deg}}$. The tabulated data also show to what extent the auxiliary curve can be manipulated by an appropriate choice of system parameters.

Case 2. $r_{ik} \neq 0$, $\alpha_{ik} = 0$.

First let us write the transfer function. In equation (4.70) we collect the terms which contain the factors $Y_{1\mathrm{ref}}$, $Y_{2\mathrm{ref}}$, $Y_{3\mathrm{ref}}$. Moreover, seeing that $\alpha_{ik} = 0$, we put A''_{11}, A''_{12}, and A''_{13} for the respective cofactors and write Δ''_3 for the system determinant.

TABLE 4.1

$$\overline{K}_{1\,\deg} = -\frac{D_1(p)\,F_{n1}(p)}{F_{1m}(p)}$$

ω	$0.01p+1$	$(2)\times 0.1p$	$(3)+1$	$(4)\times p$	$\frac{D_1(p)}{(5)+1}$	$0.3p$	$0.3p+1$	$\frac{F_{1n}(p)}{F_{1m}(p)}$	$\frac{(6)\times(9)}{-K_{\deg}}$	$K_{\deg}$
0	$1e^{j0°}$	0	$1e^{j0°}$	0	$1e^{j0°}$	0	$1e^{j0°}$	0	0	0
1	$1e^{j1°}$	$0.1e^{j91°}$	$1.01e^{j6°}$	$1.01e^{j96°}$	$1.35e^{j49°}$	$0.3e^{j90°}$	$1.05e^{j17°}$	$0.286e^{j73°}$	$0.385e^{j122°}$	$0.385e^{-j58°}$
2	$1e^{j1.5°}$	$0.2e^{j91.5°}$	$1.02e^{j11°}$	$2.04e^{j101°}$	$1.81e^{j71°}$	$0.6e^{j90°}$	$1.175e^{j31°}$	$0.51e^{j59°}$	$0.922e^{j130°}$	$0.922e^{-j50°}$
3	$1e^{j2°}$	$0.3e^{j92°}$	$1.04e^{j17°}$	$3.12e^{j107°}$	$3e^{j88°}$	$0.9e^{j90°}$	$1.35e^{j42°}$	$0.667e^{j48°}$	$2e^{j136°}$	$2e^{-j44°}$
4	$1e^{j2.5°}$	$0.4e^{j92.5°}$	$1.07e^{j22°}$	$4.28e^{j112°}$	$4e^{j98°}$	$1.2e^{j90°}$	$1.57e^{j50°}$	$0.763e^{j40°}$	$3.05e^{j138°}$	$3.05e^{-j42°}$
5	$1e^{j3°}$	$0.5e^{j93°}$	$1.1e^{j27.5°}$	$5.5e^{j117.5°}$	$5.15e^{j108°}$	$1.5e^{j90°}$	$1.81e^{j56°}$	$0.83e^{j34°}$	$4.28e^{j142°}$	$4.28e^{-j38°}$
7	$1e^{j4°}$	$0.7e^{j94°}$	$1.19e^{j36°}$	$8.33e^{j126°}$	$7.7e^{j120°}$	$2.1e^{j90°}$	$2.35e^{j64°}$	$0.893e^{j26°}$	$6.87e^{j146°}$	$6.87e^{-j34°}$
9	$1e^{j5°}$	$0.9e^{j95°}$	$1.29e^{j44°}$	$11.6e^{j134°}$	$10.9e^{j130°}$	$2.7e^{j90°}$	$2.9e^{j69°}$	$0.93e^{j21°}$	$10.1e^{j151°}$	$10.1e^{-j29°}$
10	$1.01e^{j5.5°}$	$1.01e^{j95.5°}$	$1.35e^{j48°}$	$13.5e^{j138°}$	$12.7e^{j135°}$	$3e^{j90°}$	$3.17e^{j72°}$	$0.95e^{j18°}$	$12.06e^{j153°}$	$12.06e^{-j27°}$
12	$1.015e^{j65°}$	$1.22e^{j96.5°}$	$1.49e^{j54°}$	$17.85e^{j144°}$	$17e^{j142°}$	$3.6e^{j90°}$	$3.75e^{j74°}$	$0.96e^{j16°}$	$16.3e^{j158°}$	$16.3e^{-j22°}$
15	$1.02e^{j8°}$	$1.53e^{j98°}$	$1.7e^{j63°}$	$25.5e^{j153°}$	$24.6e^{j152°}$	$4.5e^{j90°}$	$4.6e^{j78°}$	$0.98e^{j12°}$	$24.1e^{j164°}$	$24.1e^{-j16°}$
20	$1.03e^{j11°}$	$2.06e^{j101°}$	$2.1e^{j73.5°}$	$42e^{j163.5°}$	$41e^{j163°}$	$6e^{j90°}$	$6.1e^{j81°}$	$0.983e^{j9°}$	$40.2e^{j172°}$	$40.2e^{-j8°}$

TABLE 4.2

$$\overline{K}_{1\,\deg} = -\frac{D_1(p)\frac{F_{1n}(p)}{F_{1m}(p)}\left[D_2(p)\,F_{2n}(p)+K_{2\,\deg}F_{2m}(p)-F_{2n}(p)\,F_{1n}(p)\,\alpha_{12}\alpha_{21}\right]}{D_2(p)\,F_{n2}(p)+K_{2\,\deg}\,F_{2m}(p)-F_{n2}(p)\,\alpha_{21}}$$

ω	$D_1(p)$	$D_2(p)$	$\overline{K}_{1\,\deg}$	$\frac{0.2(p)}{F_{n2}(p)}$	$\frac{0.2p+1}{F_{2m}(p)}$	$\frac{(3)\times(5)}{D_2(p)F_{n2}(p)}$	$\frac{5\times(6)}{K_{2b}F_{2m}(p)}$	$\frac{0.5\times(5)}{F_{n2}(p)\alpha_{21}}$	$-F_{n2}(p)\alpha_{21}$	Denominator (10) + (14)	$\overline{K}_{1\,\deg}$ (4) : (11)	(7)+(8)
0	$1e^{j0°}$	$0.1e^{j0°}$	0	0	$1e^{j0°}$	0	$5e^{j0°}$	0	0	$5e^{j0°}$	0	$5e^{j0°}$
1	$1.35e^{j49°}$	$0.135e^{j49°}$	$0.385e^{-j58°}$	$0.2e^{j90°}$	$1.03e^{j11°}$	$0.027e^{j139°}$	$5.15e^{j11°}$	$0.1e^{j90°}$	$0.1e^{-j90°}$	$5.12e^{j10°}$	$0.075e^{-j68°}$	$5.15e^{j11°}$
2	$1.81e^{j71°}$	$0.181e^{j71°}$	$0.922e^{-j50°}$	$0.4e^{j90°}$	$1.085e^{j21.5°}$	$0.0722e^{j161°}$	$5.425e^{j21.5°}$	$0.2e^{j90°}$	$0.2e^{-j90°}$	$5.25e^{j20°}$	$0.9e^{-j40°}$	$5.37e^{j22°}$
3	$3e^{j88°}$	$0.3e^{j88°}$	$2e^{-j44°}$	$0.6e^{j90°}$	$1.175e^{j31°}$	$0.18e^{j178°}$	$5.875e^{j31°}$	$0.3e^{j90°}$	$0.3e^{-j90°}$	$5.62e^{j28°}$	$2.355e^{-j42°}$	$5.75e^{j31°}$
4	$4e^{j98°}$	$0.4e^{j98°}$	$3.05e^{-j42°}$	$0.8e^{j90°}$	$1.29e^{j41°}$	$0.32e^{j188°}$	$6.45e^{j41°}$	$0.4e^{j90°}$	$0.4e^{-j90°}$	$5.9e^{j40°}$	$3.518e^{-j42°}$	$6.2e^{j43°}$
5	$5.15e^{j108°}$	$0.515e^{j108°}$	$4.28e^{-j38°}$	$1e^{j90°}$	$1.42e^{j45°}$	$0.515e^{j198°}$	$7.1e^{j45°}$	$0.5e^{j90°}$	$0.5e^{-j90°}$	$6.4e^{j43°}$	$5.67e^{-j41°}$	$6.7e^{j46°}$
7	$7.7e^{j120°}$	$0.77e^{j120°}$	$6.87e^{-j34°}$	$1.4e^{j90°}$	$1.73e^{j55°}$	$1.08e^{j210°}$	$8.65e^{j55°}$	$0.7e^{j90°}$	$0.7e^{-j90°}$	$7.1e^{j55°}$	$7.97e^{-j49°}$	$7.65e^{j58°}$
9	$10.9e^{j130°}$	$1.09e^{j130°}$	$10.1e^{-j29°}$	$1.8e^{j90°}$	$2.07e^{j60°}$	$1.96e^{j220°}$	$10.35e^{j60°}$	$0.9e^{j90°}$	$0.9e^{-j90°}$	$7.75e^{j62°}$	$10.3e^{-j41°}$	$8.53e^{j65°}$
10	$12.7e^{j135°}$	$1.27e^{j135°}$	$12.06e^{-j27°}$	$2e^{j90°}$	$2.23e^{j63°}$	$2.54e^{j225°}$	$11.15e^{j63°}$	$1e^{j90°}$	$1e^{-j90°}$	$7.75e^{j64°}$	$12.56e^{-j41°}$	$8.7e^{j67°}$
12	$17e^{j142°}$	$1.7e^{j142°}$	$16.3e^{-j22°}$	$2.4e^{j90°}$	$2.62e^{j67°}$	$4.08e^{j232°}$	$13.1e^{j67°}$	$1.2e^{j90°}$	$1.2e^{-j90°}$	$8.2e^{j70°}$	$17.99e^{-j42°}$	$9.25e^{j73°}$
15	$24.6e^{j152°}$	$2.46e^{j152°}$	$24.1e^{-j16°}$	$3e^{j90°}$	$3.17e^{j72°}$	$7.37e^{j242°}$	$15.85e^{j72°}$	$1.5e^{j90°}$	$1.5e^{-j90°}$	$7.3e^{j77°}$	$3.3e^{-j43°}$	$8.7e^{j80°}$
20	$41e^{j163°}$	$4.1e^{j163°}$	$40.2e^{-j8°}$	$4e^{j90°}$	$4.1e^{j76°}$	$16.4e^{j253°}$	$20.5e^{j76°}$	$2e^{j90°}$	$2e^{-j90°}$	$2.4e^{j86°}$	$83.8e^{-j44°}$	$4.4e^{j88°}$

TABLE 4.3

$$K_{1\,\deg}\;\overbrace{\frac{K_{1\,\deg}}{D_2(p)\,F_{n2}(p)+K_{2b}\,F_{2m}(p)}}^{\text{(I)}}\;\overbrace{-\frac{F_{2m}(p)}{F_{1m}(p)}F_{2n}(p)\alpha_{21}}^{\text{(II)}}$$

ω	(I)	$F_{2m}(p)$	$F_{1m}(p)$	$\frac{F_{2m}(p)}{F_{1m}(p)}$	$-F_{2n}(p)\alpha_{21}$	$\frac{-11}{(5)\times(6)}$	Numerator (I) − (II) (2)+(7)	Denominator	$\frac{(8)}{(9)}$	$\frac{(10)\cdot 0.2}{(10)\cdot K_{1\,\deg}}$
0	5.0	$1e^{j0°}$	$1e^{j0°}$	$1e^{j0°}$	0	0	$5e^{j0°}$	$5e^{j0°}$	$1e^{j0°}$	$5.2e^{j0°}$
1	5.0	$1.03e^{j11°}$	$1.05e^{j17°}$	$0.98e^{-j6°}$	$0.1e^{-j90°}$	$0.098e^{-j96°}$	$5.1e^{j10°}$	$5.12e^{j10°}$	$1e^{j0°}$	$5.2e^{j0°}$
2	5.0	$1.085e^{j21.5°}$	$1.175e^{j31°}$	$0.922e^{-j9.5°}$	$0.2e^{-j90°}$	$0.184e^{-j100°}$	$5.32e^{j19°}$	$5.25e^{j20°}$	$1.01e^{j1°}$	$5.202e^{j1°}$
3	5.0	$1.175e^{j31°}$	$1.35e^{j42°}$	$0.872e^{-j11°}$	$0.3e^{-j90°}$	$0.261e^{-j100°}$	$5.6e^{j29°}$	$5.62e^{j28°}$	$1e^{j1°}$	$5.2e^{j1°}$
4	5.0	$1.29e^{j41°}$	$1.57e^{j50°}$	$0.822e^{-j9°}$	$0.4e^{-j90°}$	$0.329e^{-j99°}$	$5.9e^{j41°}$	$5.9e^{j40°}$	$1e^{j1°}$	$5.2e^{j1°}$
5	5.0	$1.42e^{j45°}$	$1.81e^{j56°}$	$0.785e^{-j11°}$	$0.5e^{-j90°}$	$0.392e^{-j101°}$	$6.4e^{j47°}$	$6.4e^{j43°}$	$1e^{j4°}$	$5.2e^{j1°}$
7	5.0	$1.73e^{j55°}$	$2.35e^{j64°}$	$0.736e^{-j9°}$	$0.7e^{-j90°}$	$0.514e^{-j99°}$	$7.2e^{j56°}$	$7.1e^{j55°}$	$1.01e^{j1°}$	$5.202e^{j1°}$
9	5.0	$2.07e^{j60°}$	$2.9e^{j69°}$	$0.715e^{-j9°}$	$0.9e^{-j90°}$	$0.642e^{-j99°}$	$7.85e^{j64°}$	$7.75e^{j62°}$	$1.01e^{j2°}$	$5.202e^{j1°}$
10	5.0	$2.23e^{j63°}$	$3.17e^{j72°}$	$0.703e^{-j9°}$	$1e^{-j90°}$	$0.703e^{-j99°}$	$8e^{j65°}$	$7.75e^{j64°}$	$1.03e^{j1°}$	$5.206e^{j1°}$

The determinant is given by

$$\Delta_3'' = \begin{vmatrix} b_1 + K_{1\,\deg}F_{1m} & K_{1\,\deg}F_{1m}r_{12} & K_{1\,\deg}F_{1m}r_{13} \\ K_{2\,\deg}F_{2m}r_{21} & b_2 + K_{2\,\deg}F_{2m} & K_{2\,\deg}F_{2m}r_{23} \\ K_{3\,\deg}F_{3m}r_{31} & K_{3\,\deg}F_{3m}r_{32} & b_3 + K_{3\,\deg}F_{3m} \end{vmatrix}; \tag{4.91}$$

to find A_{ik}'' we first write the transpose

$$A_t'' = \begin{vmatrix} b_1 + K_{1\,\deg}F_{1m} & K_{2\,\deg}F_{2m}r_{21} & K_{3\,\deg}F_{3m}r_{31} \\ K_{1\,\deg}F_{1m}r_{12} & b_2 + K_{2\,\deg}F_{2m} & K_{3\,\deg}F_{3m}r_{32} \\ K_{1\,\deg}F_{1m}r_{13} & K_{2\,\deg}F_{2m}r_{23} & b_3 + K_{3\,\deg}F_{3m} \end{vmatrix}, \tag{4.92}$$

whence

$$A_{11}'' = \begin{vmatrix} b_2 + K_{2\,\deg}F_{2m} & K_{3\,\deg}F_{3m}r_{32} \\ K_{2\,\deg}F_{2m}r_{23} & b_3 + K_{3\,\deg}F_{3m} \end{vmatrix}, \tag{4.93}$$

$$A_{12}'' = \begin{vmatrix} K_{1\,\deg}F_{1m}r_{12} & K_{3\,\deg}F_{3m}r_{32} \\ K_{1\,\deg}F_{1m}r_{13} & b_3 + K_{3\,\deg}F_{3m} \end{vmatrix}, \tag{4.93'}$$

$$A_{13}'' = \begin{vmatrix} K_{1\,\deg}F_{1m}r_{13} & b_2 + K_{2\,\deg}F_{2m} \\ K_{1\,\deg}F_{1m}r_{13} & K_{2\,\deg}F_{2m}r_{23} \end{vmatrix}. \tag{4.93''}$$

Equation (4.70) reduces to

$$\begin{aligned} Y_1 = \frac{1}{\Delta_3''}\{&[A_{11}''K_{1\,\deg}F_{1m} - A_{12}''K_{2\,\deg}F_{2m}r_{21} + A_{13}''K_{3\,\deg}F_{3m}r_{31}]\,Y_{1\,\text{ref}} + \\ &+ [A_{11}''K_{1\,\deg}F_{1m}r_{12} - A_{12}''K_{2\,\deg}F_{2m} + A_{13}''K_{3\,\deg}F_{3m}r_{32}]Y_{2\,\text{ref}} + \\ &+ [A_{11}''K_{1\,\deg}F_{1m}r_{13} - A_{12}''K_{2\,\deg}F_{2m}r_{23} + A_{13}''K_{3\,\deg}F_{3m}]Y_{3\,\text{ref}} + \\ &+ A_{11}''F_{1n}f_1 - A_{12}'F_{2n}f_2 + A_{13}'F_{3n}f_3\}. \end{aligned} \tag{4.94}$$

The transfer function (ignoring the load) is written as

$$\frac{Y_1(p)}{Y_{1\text{ref}}(p)} = \frac{A''_{11}K_{1\,\text{deg}}F_{1m}(p) - A''_{12}K_{2\,\text{deg}}F_{2m}(p)\,r_{21} + A''_{13}K_{3\,\text{deg}}F_{3m}(p)\,r_{31}}{\Delta''_3} +$$
$$+ \frac{A''_{11}K_{1\,\text{deg}}F_{1m}(p)\,r_{12} - A''_{12}K_{2\,\text{deg}}F_{2m}(p) + A''_{13}K_{3\,\text{deg}}F_{3m}(p)\,r_{32}}{\Delta''_3} \cdot \frac{Y_{2\,\text{ref}}(p)}{Y_{1\,\text{ref}}(p)} +$$
$$+ \frac{A''_{11}K_{1\,\text{deg}}F_{1m}(p)\,r_{13} - A''_{12}K_{2\,\text{deg}}F_{2m}(p)\,r_{23} + A''_{13}K_{3\,\text{deg}}F_{3m}(p)}{\Delta''_3} \cdot \frac{Y_{3\,\text{ref}}(p)}{Y_{1\,\text{ref}}(p)}. \qquad (4.95)$$

Transposing the determinant (4.91) and expanding the transpose in elements of the first row, we find

$$\Delta''_3 = (b_1 + K_{1\,\text{deg}}F_{1m})\,A''_{11} - K_{2\,\text{deg}}F_{2m}r_{21}A''_{12} + K_{3\,\text{deg}}F_{3m}r_{31}A''_{13}. \qquad (4.96)$$

The elements of the first columns in A''_{12} and A''_{13} are multiplied by $K_{1\,\text{deg}}$. Taking this factor outside the determinant, we write

$$A''_{12} = K_{1\,\text{deg}}A_{12}, \quad A''_{13} = K_{1\,\text{deg}}A_{13}. \qquad (4.97)$$

Making use of (4.96) and (4.97), we write the transfer function (4.95) in the form

$$\frac{Y_1(p)}{Y_{1\,\text{ref}}(p)} = \frac{K_{1\,\text{deg}}\left[A''_{11}(p)\,F_{1m}(p) - K_{2\,\text{deg}}F_{2m}(p)\,r_{21}A_{12}(p) + K_{3\,\text{deg}}F_{3m}(p)\,r_{31}A_{13}(p)\right]}{b_1A''_{11}(p) + K_{1\,\text{deg}}\left[F_{1m}(p)\,A''_{11}(p) - K_{2\,\text{deg}}F_{2m}(p)\,r_{21}A_{12}(p) + \rightarrow\right.} +$$
$$\left. \rightarrow + K_{3\,\text{deg}}F_{3n}(p)\,r_{31}A_{13}(p)\right]$$
$$+ \frac{Y_{2\,\text{ref}}(p)}{Y_{1\,\text{ref}}(p)} \times \frac{K_{1\,\text{deg}}\left[A''_{11}(p)\,F_{1m}(p)\,r_{12} - K_{2\,\text{deg}}F_{2m}(p)\,A_{12}(p) + K_{3\,\text{deg}}F_{3m}(p)\,r_{32}A_{13}(p)\right]}{b_1A''_{11}(p) + K_{1\,\text{deg}}\left[F_{1m}(p)\,A''_{11}(p) - K_{2\,\text{deg}}F_{2m}(p)\,r_{21}A_{12}(p) + \rightarrow\right.} +$$
$$\left. \rightarrow + K_{3\,\text{deg}}F_{3m}(p)\,r_{31}A_{13}(p)\right]$$
$$+ \frac{Y_{3\,\text{ref}}(p)}{Y_{1\,\text{ref}}(p)} \times \frac{K_{1\,\text{deg}}\left[A''_{11}(p)\,F_{1m}(p)\,r_{13} - K_{2\,\text{deg}}F_{2m}(p)\,r_{23}A_{12}(p) + A_{13}(p)\,K_{3\,\text{deg}}F_{3m}(p)\right]}{b_1A''_{11}(p) + K_{1\,\text{deg}}\left[F_{1m}(p)\,A''_{11}(p) - K_{2\,\text{deg}}F_{2m}(p)\,r_{21}A_{12}(p) + \rightarrow\right.} \qquad (4.98)$$
$$\left. \rightarrow + K_{3\,\text{deg}}F_{3m}(p)\,r_{31}A_{13}(p)\right]$$

Dividing the numerator and the denominator of (4.98) by

$$F_{1m}(p)\,A''_{11}(p) - K_{2\,\text{deg}}F_{2m}(p)\,r_{21}A_{12}(p) + K_{3\,\text{deg}}F_{3m}(p)\,r_{31}A_{13}(p)$$

and putting $p = j\omega$, we find

$$\frac{Y_1(j\omega)}{Y_{1\,\text{ref}}(j\omega)} = \frac{K_{1\,\text{deg}}}{K_{1\,\text{deg}} + D_{K_1}(j\omega)} + \frac{K_{1\,\text{deg}}\dfrac{P_1}{Q_1}}{K_{1\,\text{deg}} + D_{K_1}(j\omega)} \cdot \frac{Y_{2\,\text{ref}}(j\omega)}{Y_{1\,\text{ref}}(j\omega)} + \frac{K_{1\,\text{deg}}\dfrac{P_2}{Q_2}}{K_{1\,\text{deg}} + D_{K_1}(j\omega)} \cdot \frac{Y_{1\,\text{ref}}(j\omega)}{Y_{1\,\text{ref}}(j\omega)}, \qquad (4.99)$$

where

$$P_1 = A''_{11}(j\omega)\,F_{1m}(j\omega)\,r_{12} - K_{2\,\text{deg}}F_{2m}(j\omega)\,A_{12}(j\omega) +$$
$$+ K_{3\,\text{deg}}F_{3m}(j\omega)\,A_{13}(j\omega)\,r_{32},$$
$$Q_1 = F_{1m}(j\omega)\,A''_{11}(j\omega) - K_{2\,\text{deg}}F_{2m}(j\omega)\,r_{21}A_{12}(j\omega) +$$
$$+ K_{3\,\text{deg}}F_{3m}(j\omega)\,r_{13}A_{13}(j\omega),$$
$$P_2 = A''_{11}(j\omega)\,F_{1m}(j\omega)\,r_{13} - K_{2\,\text{deg}}F_{2m}(j\omega)\,r_{23}A_{12}(j\omega) +$$
$$+ K_{3\,\text{deg}}F_{3m}(j\omega)\,A_{13}(j\omega),$$
$$Q_2 = F_{1m}(j\omega)\,A''_{11}(j\omega) - K_{2\,\text{deg}}F_{2m}(j\omega)\,r_{21}A_{12}(j\omega) +$$
$$+ K_{3\,\text{deg}}F_{3m}(j\omega)\,r_{13}A_{13}(j\omega).$$

Here $D_{K_1}(j\omega)$ is the equation of the D-decomposition curve for $K_{1\,\mathrm{deg}}$ with its sign reversed, defined by the equation

$$\overline{K}_{1\,\mathrm{deg}} = -\frac{b_1(j\omega)\,A''_{11}(j\omega)}{F_{1m}(j\omega)\,A''_{11}(j\omega) - K_{2\,\mathrm{deg}}F_{2m}(j\omega)\,r_{21}A_{12}(j\omega) + K_{3\,\mathrm{deg}}F_{3m}(j\omega)\,r_{31}A_{13}(j\omega)}. \qquad (4.100)$$

The first term in the right-hand side of (4.99) determines the dynamic properties of an isolated servosystem. These properties can be found from the D-decomposition curve (4.100). Subsequent terms specify the influence of other servosystems on the one being considered. Since the system is linear, the effect of extraneous servosystems can be found by superposition. All terms in the right-hand side of (4.99) have a common denominator, and the D-decomposition curve $D_{K_i}(j\omega)$ is thus applicable to all the components. It therefore suffices to perform geometrical addition of the auxiliary curves only.

In the general case of an n-dimensional servosystem, the dynamics of the i-th servo is found from the general equation

$$\frac{Y_i(j\omega)}{Y_{i\,\mathrm{ref}}(j\omega)} = \frac{K_{i\,\mathrm{deg}}}{K_{i\,\mathrm{deg}} + D_{K_i}(j\omega)}\left[1 + \sum_{k=1}^{n}\xi\left[A_{ik}(j\omega)\,r_{ik}\right]\right]\frac{Y_{k\,\mathrm{ref}}(j\omega)}{Y_{i\,\mathrm{ref}}(j\omega)}. \qquad (4.101)$$

The functions $\xi[A_{ik}(j\omega)r_{ik}]$ are obtained as previously for a three-dimensional servo. The expression in brackets in (4.101) is the auxiliary curve for the general case of an n-dimensional servosystem. Having constructed the D-decomposition curve for $K_{i\,\mathrm{deg}}$ and the auxiliary curve, we can choose the appropriate gain $K_{i\,\mathrm{deg}}$ which ensures system stability and desired quality.

Some general conclusions concerning the dynamics of this class of structures can be drawn from (4.99) and (4.101).

1. The auxiliary curve, representing the contribution from extraneous servosystems, may raise the crossover frequency of a closed-loop i-th servo at constant gain. This is obvious from Figure 4.4, where ω_{10} is the crossover frequency of gain in an uncoupled system, ω_{20} the crossover frequency for the same gain $K_{1\,\mathrm{deg}}$ in a system with the auxiliary curve shown in the figure. Hence follows a very important conclusion: the dynamic properties of each i-th servo in a multidimensional servosystem can be better than those of an isolated i-th servo.

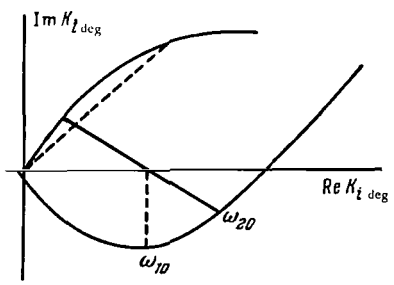

FIGURE 4.4. Estimating the crossover frequency.

2. The dynamic properties of each component servo can be adjusted by an appropriate choice of variation of the reference values $Y_{k\,\mathrm{ref}}$. We see from (4.101) that the auxiliary curve of the i-th servo is substantially dependent on the variation of $Y_{k\,\mathrm{ref}}$ of all the other servos. This in a sense provides a sort of control coupling, and in certain cases a sequence of $Y_{k\,\mathrm{ref}}$ values can be programmed in advance to ensure the desired quality characteristics of the i-th controlled variable.

As an example, we calculate a two-dimensional servosystem, which illustrates the procedure and also validates the above conclusions.

From (4.99) we have for the transfer function of a two-dimensional system

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{K_{1\,\deg}}{K_{1\,\deg} + D_{K_1}(p)} + \frac{K_{1\,\deg}\dfrac{A''_{11}F_{1m}(p)\,r_{12} - K_{2\,\deg}F_{2m}(p)\,A''_{12}(p)}{F_{1m}(p)\,A''_{11}(p) - K_{2\,\deg}F_{2m}(p)\,r_{21}A''_{12}(p)}}{K_{1\,\deg} + D_{K_1}(p)} \cdot \frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}.$$

Here

$$A''_{11} = b_2 + K_{2\,\deg}F_{2m},$$
$$A''_{12} = K_{1\,\deg}F_{1m}r_{12},$$
$$D_{K_1}(p) = \frac{D_1(p)\,F_{1n}(p)\,[b_1(p) + K_{2\,\deg}F_{2m}(p)]}{F_{1m}(p)\,[b_2(p) + K_{2\,\deg}F_{2m}(p)] - K_{2\,\deg}F_{2n}(p)\,r_{21}F_{1m}(p)\,r_{12}},$$
$$p = j\omega$$
$$b_1 = a_0p^3 + a_1p^2 + a_2p + a_3,$$
$$b_2 = a'_0p^3 + a'_1p^2 + a'_2p + a'_3,$$
$$a_0 = 0.001, \quad a_1 = 0.1, \quad a_2 = 1, \quad a_3 = 1,$$
$$a'_0 = 0.0001, \quad a'_1 = 0.001, \quad a'_2 = 0.1, \quad a'_3 = 0.1,$$
$$r_{12} = r_{21} = 0.2,$$
$$\alpha_{12} = \alpha_{21} = 0.5.$$

Figure 4.5 plots (a) the D-decomposition curve for $r_{12}=r_{21}=0.2$, (b) the D-decomposition curve for $r_{12}=r_{21}=0.5$. Figure 4.6 shows separately the auxiliary curves for $r_{12}=0.2$ and $r_{12}=0.5$.

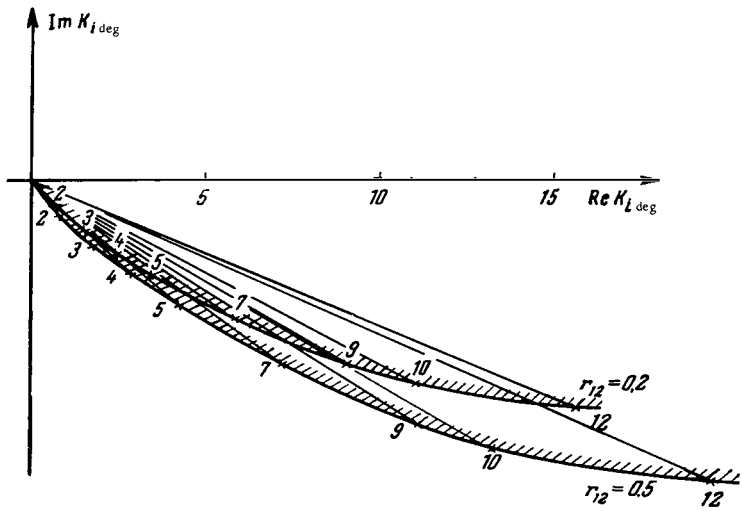

FIGURE 4.5. D-decomposition curve.

The various curves indicate that as the degree of coupling increases, the D-decomposition curve becomes more favorable: the range of $K_{1\,\mathrm{deg}}$ values corresponding to a stable system increases, and the crossover frequency for the same $K_{1\,\mathrm{deg}}$ is higher. Furthermore, in a coupled system, the auxiliary curve can be modified by appropriately changing $Y_{i\,\mathrm{ref}}$.

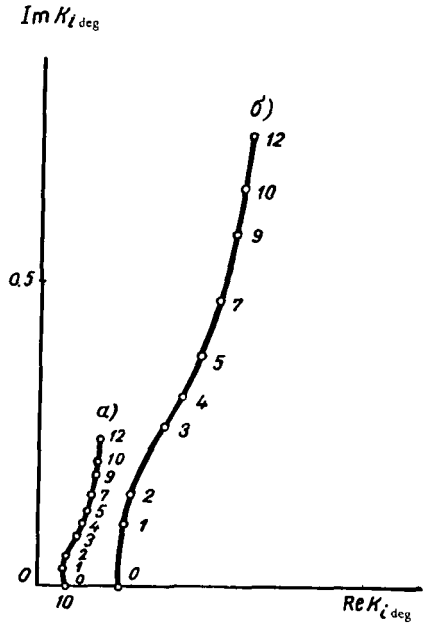

FIGURE 4.6. Auxiliary curves:

(a) $r_{12}=0.2$, (b) $r_{12}=0.5$.

Case 3. The general case $r_{ik}\neq 0$ and $\alpha_{ik}\neq 0$.

System calculations and choice of parameters in accordance with quality specifications can be divided into two separate stages, putting first $r_{ik}\neq 0$, $\alpha_{ik}=0$ and then $r_{ik}=0$, $\alpha_{ik}\neq 0$, and adding the results. The parameters are chosen so as to ensure the desired system dynamics with a view to the task at hand (designing a servosystem, stabilizing, etc.).

(b) INTEGRAL SYSTEMS

We shall establish how the expression for the j-th controlled variable changes when integrating control is introduced in each loop and derive working formulas for system analysis and choice of fundamental parameters.

We have previously obtained an expression for the j-th controlled variable in an integral multivariable system. This expression is

$$Y_j(p)=\frac{1}{\Delta}\left\{(-1)^{\rho+j}A_{\rho j}(p)\left[K_{\rho\,\mathrm{tot}}F_{m\rho}(p)\sum_{k=1}^{n}r_{\rho k}Y_{k\,\mathrm{ref}}(p)+\right.\right.$$
$$\left.\left.+\left(c_\rho(p)+K_\rho K_{\mathrm{st}}\,g_\rho(p)\right)f_\rho(p)\right]\right\}. \qquad (4.50)$$

As in the case of proportional systems, we assume that the configuration remains structurally stable as the gain increases indefinitely. In other words, condition (4.2) is again satisfied. Stabilization is provided by passive elements meeting condition (4.55).

We now derive an equation for the j-th controlled variable assuming sufficiently large gain for the stabilized section. Dividing (4.42) by $K_{i\,\mathrm{st}}$ and putting $\frac{1}{K_{i\,\mathrm{st}}} = m_i$ we obtain (making use of nomenclature (4.41))

$$m_i a_i(p) Y_i(p) + b_i(p) Y_i(p) + K_{i\,\mathrm{deg}} F_{mi}(p) Y_i(p) + \\ + K_{i\,\mathrm{deg}} F_{mi}(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} r_{ik} Y_k(p) + m_i c_i(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k(p) + \\ + K_i g_i(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k(p) = K_{i\,\mathrm{deg}} F_{mi}(p) Y_{i\,\mathrm{ref}}(p) + \\ + K_{i\,\mathrm{deg}} F_{mi}(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} r_{ik} Y_{k\,\mathrm{ref}}(p) + m_i c_i(p) f_i(p) + K_i g_i(p) f_i(p) \\ \left(i = 1, 2, \ldots, n \qquad K_{i\,\mathrm{deg}} = \frac{K_{i\,\mathrm{tot}}}{K_{i\,\mathrm{st}}}\right). \tag{4.102}$$

Let the auxiliary equation (which may be of first, second, or third kind in this case) satisfy the stability conditions. It thus suffices to ensure stability of the degenerate equation and to choose its parameters in compliance with system quality specifications. The point is, that in a stable system with sufficiently large $K_{i\,\mathrm{st}}$, the quality of the entire system is completely determined by the degenerate equation.

The set of degenerate equations is derived from (4.102) by putting $m_i = 0$. We have

$$[b_i(p) + K_{i\,\mathrm{deg}} F_{mi}(p)] Y_i(p) + K_{1\,\mathrm{deg}} F_{mi}(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} r_{ik} Y_k(p) + \\ + K_i g_i(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k(p) = K_{i\,\mathrm{deg}} F_{mi}(p) Y_{i\,\mathrm{ref}}(p) + \\ + K_{i\,\mathrm{deg}} F_{im}(p) \sum_{\substack{k=1 \\ k \neq i}}^{n} r_{ik}(p) Y_{k\,\mathrm{ref}}(p) + K_i g_i(p) f_i(p) \\ (i = 1, 2, \ldots, n). \tag{4.103}$$

In matrix form equations (4.103) are written as follows:

$$AY = K_{\mathrm{deg}} F_m Y_{\mathrm{ref}} + BY_{\mathrm{ref}} + NF, \tag{4.104}$$

whence

$$Y = A^{-1} |K_{\mathrm{deg}} F_m Y_{\mathrm{ref}} + BY_{\mathrm{ref}} + NF|, \tag{4.105}$$

where

$$A = \begin{Vmatrix} b_1(p) + K_{1\,\mathrm{deg}} F_{m1}(p) & K_{1\,\mathrm{deg}} F_{1m}(p) r_{12} + \ldots & K_{1\,\mathrm{deg}} F_{m1}(p) r_{1n} + K_1 g_1(p) \alpha_{1n} \\ K_{2\,\mathrm{deg}} F_{2m}(p) r_{21} + K_2 g_2(p) \alpha_{21} & b_2(p) + K_{2\,\mathrm{deg}} F_{m2}(p) \ldots & K_{2\,\mathrm{deg}} F_{m2}(p) r_{2n} + K_2 g_2(p) \alpha_{2n} \\ \ldots & \ldots & \ldots \\ K_{n\,\mathrm{deg}} F_{nm}(p) r_{n1} + K_n g_n(p) \alpha_{n1} & \ldots & \ldots b_2(p) + K_{2\,\mathrm{deg}} F_{mn}(p) \end{Vmatrix}, \tag{4.106}$$

$$K_{\deg}F_m=\begin{Vmatrix} K_{1\deg}F_{m1}(p) & 0 & \dots & 0 \\ 0 & K_{2\deg}F_{m2}(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & K_{n\deg}F_{m2}(p) \end{Vmatrix}, \tag{4.107}$$

$$Y_{\text{ref}}=\begin{Vmatrix} Y_{1\,\text{ref}}(p) \\ Y_{2\,\text{ref}}(p) \\ \cdot \\ \cdot \\ \cdot \\ Y_{n\,\text{ref}}(p) \end{Vmatrix}, \tag{4.108}$$

$$B=\begin{Vmatrix} 0 & K_{1\deg}F_{m1}(p)\,r_{12} & \dots & K_{1\deg}F_{m1}(p)\,r_{1n} \\ K_{2\deg}F_{m2}(p)\,r_{21} & 0 & \dots & K_{2\deg}F_{m2}(p)\,r_{2n} \\ \dots & \dots & \dots & \dots \\ K_{n\deg}F_{mn}(p)\,r_{n1} & \dots & \dots & 0 \end{Vmatrix}, \tag{4.109}$$

$$N=\begin{Vmatrix} K_1g_1(p) & 0 & \dots & 0 \\ 0 & K_2g_2(p) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & K_ng_n(p) \end{Vmatrix}, \tag{4.110}$$

and

$$F=\begin{Vmatrix} f_1(p) \\ f_2(p) \\ \cdot \\ \cdot \\ \cdot \\ f_n(p) \end{Vmatrix}. \tag{4.111}$$

The inverse A^{-1} is obtained by the previously outlined method from the transpose A_i. Inserting for the matrices in (4.105) their expressions (4.106)—(4.111) and multiplying, we find

$$Y=\frac{1}{\Delta}\begin{Vmatrix} \sum_{j=1}^{n}(-1)^{1+j}A_{1j}\left(K_jF_{mj}\sum_{k=1}^{n}r_{jk}Y_{k\,\text{ref}}+K_1g_1f_1\right) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{j=1}^{n}(-1)^{n+j}A_{nj}\left(K_jF_{mj}\sum_{k=1}^{n}r_{jk}Y_{k\,\text{ref}}+K_ng_nf_n\right) \end{Vmatrix}, \tag{4.112}$$

where all $r_{ii}=1$.

Hence for the j-th controlled variable

$$Y_j=\frac{1}{\Delta}\sum_{i=1}^{n}(-1)^{j+i}A_{ji}\left(K_iF_{mi}\sum_{k=1}^{n}r_{ik}Y_{\text{ref}\,k}+K_jg_jf_j\right). \tag{4.113}$$

The structure of (4.113) is identical to that of the equation of the j-th controlled variable in a proportional system. The only difference is in the explicit expressions of the operators in (4.69) and (4.113). It is thus unnecessary to repeat the previous manipulations described in detail for proportional control systems. Integral systems can now be investigated and calculated using equation (4.80) with appropriate expressions inserted for the operators from (4.113).

As an example, we proceed with a calculation of a two-variable integral control system. Here $\alpha_{ik}\neq 0$, $r_{ik}=0$; we thus start with working formula (4.83). Here

$$K_{1\deg}=\frac{K_{1\deg}-\dfrac{F_{2m}(p)}{F_{1m}(p)}\dfrac{A'_{12}(p)}{A'_{11}(p)}\dfrac{Y_{2\,\text{ref}}(p)}{Y_{1\,\text{ref}}(p)}}{K_{1\deg}+\dfrac{b_1(p)\,A'_{11}(p)-F_{2n}(p)\,A'_{12}(p)\,\alpha_{21}}{F_{1m}(p)\,A'_{11}(p)}}. \tag{4.83}$$

Remembering that we are dealing with an integral system, we will determine the expressions for all the operators entering (4.83). For simplicity a lagless system is assumed, and making use of nomenclature (4.41) we obtain in the integrating case

$$b_i = R_i(p)\,pD_i(p)\,F_{ni}(p),$$
$$A'_{11} = b_2(p) + K_{2\,\deg} F_{2m}(p) = R_2(p)\,pD_2(p)\,F_{n2}(p) + K_{2\,\deg} F_{2m}(p),$$
$$A'_{12} = -F_{2n}\alpha_{21}.$$

Substituting in (4.83), we find

$$K_{1\,\deg} = \frac{L(p) - N(p)}{\varphi(p)},$$

where

$$L(p) = K_{1\,\deg},$$
$$N(p) = -\frac{F_{2m}(p)}{F_{1m}(p)} \cdot \frac{F_{2n}(p)\,\alpha_{21}}{R_2(p)\,D_2(p)\,pF_{n2}(p) + K_{2\,\deg} F_{2m}(p)} \cdot \frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)},$$
$$\varphi(p) = K_{1\,\deg} + \frac{R_1(p)\,pD_1(p)\,F_{n1}(p) - F_{2n}(j\omega)\,\alpha_{12}}{F_{1m}(p)\,[R_2(p)\,D_2(p)\,F_{n2}(p)\,p + K_{2\,\deg} F_{2m}(p)}.$$

The equation of the D-decomposition curve for $K_{1\,\deg}$ is

$$\bar{K}_{1\,\deg} = -\frac{R_1(j\omega)\,D_1(j\omega)\,F_{n1}(j\omega)\,j\omega - F_{2n}(j\omega)\,\alpha_{12}}{F_{1m}(j\omega)\,[R_2(j\omega)\,D_2(j\omega)\,F_{n2}(j\omega)\,j\omega + K_{2\,\deg} F_{2m}(j\omega)}.$$

For the sake of simplicity we put

$$R_i(j\omega) = 1,$$
$$F_{im} = 1 + \tau_i p,$$
$$F_{in} = \tau_i p.$$

The calculations are then continued as for a proportional control system.

§ 4.3. STRUCTURES WITH SEVERAL STABILIZERS

Stabilizers using passive elements have the obvious advantage that technically their design and construction involve neither fundamental nor practical difficulties. On the other hand, it is clear from the preceding and from the very nature of the passive elements that

$$n_i - m_i \leqslant 0,$$

so that the self-operator of the stabilized section of the loop cannot be of degree higher than two.

It is shown in /39/ that in single-variable control systems a single stabilizer, though possibly ensuring infinite-gain stability, is insufficient for high-quality operation. This is so because the degenerate equation is of a high degree and the dynamic properties of the system are inadequate.

Systems of rational structure considered in /39/ possess infinite-gain stability and an unlimited closed-loop positive-response bandwidth. In cases when the initial single-loop system is described by an equation of higher than fourth degree, the desired structure is generated by introducing several stabilizers using passive elements.

In this section we generalize the preceding results to the case of ν stabilizers in the system. Owing to the inclusion of numerous stabilizers, the system now constitutes a multiloop structure in each controlled variable.

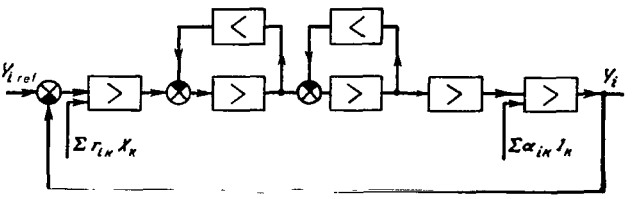

FIGURE 4.7. A multiloop subsystem.

Figure 4.7 is a block diagram of the layout for the i-th controlled variable. In the derivation of the general equation we allow for coupling through the plant and the measuring devices. By putting subsequently $r_{ik} = 0$, we will obtain the equation of an ordinary plant-coupled multivariable system.

We assume that the ν elements whose gain can be made sufficiently large are stabilized; part of the measurement device, part of the controller, and the plant are not stabilized. The set of equations describing the behavior of the i-th controlled variable in this system is the following.

The plant equation:

$$D_i(p)\,Y_i = K_i\left[Y_i' - \sum_{\substack{k=1\\k\neq i}}^{n} a_{ik}Y_k + f_i\right]. \tag{4.114}$$

The equation of the unstabilized part of the controller:

$$R_i(p)\,Y_i' = \mu_i X_{iv}. \tag{4.115}$$

The equations of the ν stabilized elements in No. 1 configuration /39/:

$$\prod_{\rho=1}^{\nu}\left[N_{i\rho}F_{\rho mi} + K_{i\rho}F_{ni\rho}\right]X_\nu = \prod_{\rho=1}^{\nu} K_{i\rho}F_{mi\rho}X_{i1}. \tag{4.116}$$

The equation of the unstabilized part of the measurement device:

$$Q_i(p)\,X_{1i} = \delta_i\left[Y_{i\,\mathrm{ref}} - Y_i + \sum_{\substack{k=1\\k\neq i}}^{n} r_{ik}\left(Y_{k\,\mathrm{ref}} - Y_k\right)\right]. \tag{4.117}$$

Eliminating Y_l', X_{lv}, and X_{ln} between (4.114), (4.115), (4.116), and (4.117), we obtain after simple manipulations

$$\left\{R_i(p)\,D_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]+\right.$$

$$\left.+K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{mi}(p)\right\}Y_i+R_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+$$

$$+K_{i\rho}F_{ni\rho}(p)]\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}Y_k+\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{mi\rho}\sum_{\substack{k=1\\k\neq l}}^{n}r_{ik}Y_k=$$

$$=K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{mi\rho}(p)\,Y_{i\,\mathrm{ref}}+$$

$$+K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{mi\rho}(p)\sum_{\substack{k=1\\k\neq l}}^{n}r_{ik}Y_{k\,\mathrm{ref}}+$$

$$+R_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]\,f_i. \qquad (4.118)$$

Putting $i=1, 2, \ldots,$ we obtain the complete set of equations describing this multivariable control system.

An ordinary plant-coupled multivariable system is obtained by putting $r_{ik}=0$ in (4.118). Thus

$$\left\{R_i(p)\,D_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]+\right.$$

$$\left.+K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{mi\rho}(p)\right\}Y_i+$$

$$+R_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}Y_k=$$

$$=K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}F_{im\rho}Y_{i\,\mathrm{ref}}+$$

$$+R_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{im\rho}(p)+K_{i\rho}F_{ni\rho}(p)]\,f_i. \qquad (4.119)$$

The following notation will be needed if we are to write (4.118) in matrix form:

$$R_i(p)\,D_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]=a_i(p),$$

$$a_{ii}=a_i(p)+K_{i\,\mathrm{tot}}\prod_{\rho=1}^{v}F_{mi\rho}(p),$$

$$\prod_{\rho=1}^{v}F_{m\rho i}(p)=\zeta_i(p),$$

$$K_i\mu_i\delta_i\prod_{\rho=1}^{v}K_{i\rho}=K_{i\,\mathrm{tot}},$$

$$R_i(p)\,Q_i(p)\prod_{\rho=1}^{v}[N_{i\rho}(p)\,F_{mi\rho}(p)+K_{i\rho}F_{ni\rho}(p)]=b_i(p).$$

Equations (4.118) are thus rewritten as

$$\left[a_i(p)+K_{i\,\mathrm{tot}}\prod_{\rho=1}^{v}F_{mi\rho}(p)\right]Y_i+b_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}Y_k=$$

$$=K_{i\,\mathrm{tot}}\prod_{\rho=1}^{v}F_{mi\rho}(p)\,Y_{i\,\mathrm{ref}}+b_i(p)\,f_i \qquad (i=1, 2, \ldots, n). \qquad (4.120)$$

The matrix form of (4.119) is thus

$$AY = K_{\text{tot}}\zeta Y_{\text{ref}} + BF, \tag{4.121}$$

whence

$$Y = A^{-1}[K_{\text{tot}}\zeta Y_{\text{ref}} + BF], \tag{4.122}$$

where

$$A = \begin{Vmatrix} a_{11} & b_1a_{12} & \dots & b_1a_{1n} \\ b_2a_{21} & a_{22} & \dots & b_2a_{2n} \\ \dots & \dots & \dots & \dots \\ b_ja_{j1} & & a_{jj} & b_ja_{jn} \\ \dots & \dots & \dots & \dots \\ b_na_{n1} & & \dots & a_{nn} \end{Vmatrix},$$

$$K_{\text{tot}}\zeta Y_{\text{ref}} = \begin{Vmatrix} K_{1\,\text{tot}}\zeta_1 Y_{1\,\text{ref}} \\ \vdots \\ K_{n\,\text{tot}}\zeta_n Y_{n\,\text{ref}} \end{Vmatrix}, \qquad BF = \begin{Vmatrix} b_1f_1 \\ \vdots \\ b_nf_n \end{Vmatrix}.$$

The inverse A^{-1} is calculated as before.

Since equation (4.122) has the same form as equation (4.121), we can directly write the equation for the j-th controlled variable:

$$Y_j(p) = \frac{1}{\Delta}\left\{\sum_{\rho=1}^{n}(-1)^{j+\rho}A_{j\rho}\left[K_{\rho\,\text{tot}}\prod_{k=1}^{\nu}F_{m\rho k}(p)Y_{\rho\,\text{ref}} + b_\rho(p)f_\rho\right]\right\}.$$

Assuming that all $Y_{\rho\,\text{ref}}$ (with the exception of $\rho = j$) are known numerical values and that $Y_{j\,\text{ref}}$ is the input, we obtain the following expression for the transfer function for the j-th controlled variable (taking $f_\rho = 0$):

$$\begin{aligned}\frac{Y_j(p)}{Y_{j\,\text{ref}}} &= \frac{1}{\Delta}A_{jj}(p)K_{j\,\text{tot}}\prod_{k=1}^{\nu}F_{mjk}(p) + \\ &+ \frac{1}{\Delta Y_{j\,\text{ref}}}\sum_{\substack{\rho=1\\ \rho\neq j}}^{n}(-1)^{j+\rho}A_{j\rho}\left[K_{\rho\,\text{tot}}\prod_{k=1}^{\nu}F_{m\rho k}(p)Y_{k\,\text{ref}}\right] = \\ &= \frac{1}{\Delta}\left\{A_{jj}(p)\prod_{k=1}^{\nu}F_{mjk}K_{j\,\text{tot}} + \right. \\ &\left. + \frac{1}{Y_{j\,\text{ref}}}\sum_{\substack{\rho=1\\ \rho\neq j}}^{n}(-1)^{j+\rho}A_{j\rho}K_{\rho\,\text{tot}}\prod_{k=1}^{\nu}F_{m\rho k}(p)Y_{k\,\text{ref}}\right\}.\end{aligned} \tag{4.123}$$

This generalization can be interpreted as follows. Since the stabilizers use passive elements where the degree of p in the numerator (n_i) is invariably less than or equal to the degree of p in the denominator (m_i), i.e., $n_i \leqslant m_i$, the stabilized section in a system with a single stabilizer can be described by a differential equation of not higher than second degree. Our generalization lifts this essential restriction. It is proved that the stabilized section can be described by a differential equation of any degree, provided that not one but n stabilizers are introduced. The number of stabilizers n depends on the degree of the equation describing the stabilized section. If the degree of this equation is ν, the minimum number of passive-element stabilizers for this loop is $n = \frac{\nu}{2}$.

Chapter Five

COMBINED MULTIVARIABLE CONTROL SYSTEMS

§ 5.1. INTRODUCTORY REMARKS

Combined multivariable control systems are automatic control systems with plant and load coupling between the various controlled variables. All loads act as disturbances on all controlled variables.

The present analysis of combined control systems is based on two principles, the Watt—Polzunov principle (or the principle of control by deviation) and the principle of load control.

A simple problem to be considered at the outset is the choice of rational structure. In ordinary plant-coupled multivariable systems the choice of structure reduces to the determination of stabilizer properties and points of stabilizer connection to the network that meet certain quality and functional specifications. In combined control systems one is additionally concerned with the transducer through which disturbances are introduced into the control loop and with the connection of its output to the system.

Aside from the requirements for ordinary multivariable control systems, we should consider certain invariance aspects of the structural properties of these systems. Invariance is dealt with in a special chapter.

§ 5.2. TRANSFER FUNCTIONS

Figure 5.1 is a block diagram of a combined control system. No restrictions are imposed on the elements. The stabilizer $\frac{F_{ni}(p)}{F_{mi}(p)}$ is chosen so that K_{ci} may increase indefinitely. In general, it follows from the results of Chapter Four that if $\frac{K_{ci}}{R_i(p)}$ is such that the stabilizer should have $n_i > m_i$, $R_i(p)$ can be structurally partitioned and several stabilizers introduced; this approach will not affect the fundamental results. For this reason the transfer functions $\frac{K_{ci}}{R_i(p)}$ and $\frac{F_{ni}(p)}{F_{mi}(p)}$ are structurally of very general character.

So as not to restrict the generality of our analysis, a section of the loop with a transfer function $\frac{\mu_i}{Q_i(p)}$ is left unstabilized. No restriction is imposed on this transfer function at the present stage, but later on it will

turn out that the parameters of the unstabilized element should be chosen so that the degenerate equation remain stable as $K_{ci} \to \infty$. The sum of all load disturbances applied to the plant constitutes the input of this element.

The transfer function $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ is unknown at this stage, and it is therefore immaterial at what particular point of the main control loop the output of the transducer $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ is delivered.

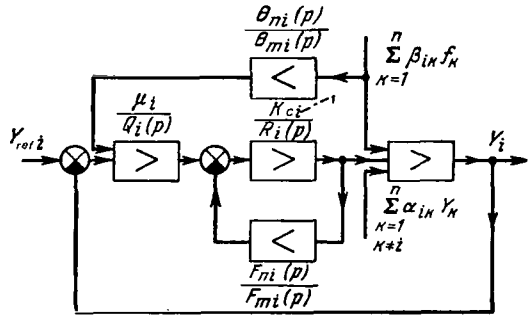

FIGURE 5.1. A combined control system.

Our problem is the following: given a certain quality criterion or a certain desirable property of the combined control system, choose the transfer function $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ in compliance with the properties of the section between the transducer output and the plant input, where the load disturbances are applied. Once the sought property of the transducer has been determined and its transfer function established, the connection of the output can be found unambiguously.

The transfer function for the i-th controlled variable, according to Figure 5.1, is

$$Q_i(p) X_i = \mu_i \left[Y_{ref\,i} - Y_i + \frac{\theta_{ni}(p)}{\theta_{mi}(p)} \sum_{k=1}^{n} \beta_{ik}(p) f_k \right]$$

or

$$Q_i(p)\theta_{mi}(p) X_i = \mu_i \left[(Y_{ref\,i} - Y_i)\theta_{mi}(p) + \theta_{ni}(p) \sum_{k=1}^{n} \beta_{ik}(p) f_k \right],$$

$$R_i(p) Y_i' = K_{ci} \left[X_i - \frac{F_{ni}(p)}{F_{mi}(p)} Y_i' \right] \qquad (5.1)$$

i.e.,

$$[R_i(p) F_{mi}(p) + K_{ci} F_{ni}(p)] Y_i' = K_{ci} F_{mi}(p) X_i, \qquad (5.2)$$

$$D_i(p) Y_i = K_i \left[Y_i' - \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik}(p) Y_k + \sum_{k=1}^{n} \beta_{ik}(p) f_k \right]. \qquad (5.3)$$

Eliminating X_i, Y_i' between (5.1), (5.2), and (5.3), we find

$$Q_i(p)\theta_{mi}(p)[R_i(p)F_{mi}(p)+K_{ci}F_{ni}(p)]Y_i'=\mu_iK_{ci}F_{mi}(p)\left[(Y_{refi}-Y_i)\theta_{mi}(p)+\theta_{ni}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k\right],$$

and

$$Y_i'=\frac{D_i(p)}{K_i}Y_i+\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k-\sum_{k=1}^{n}\beta_{ik}(p)f_k$$

$$Q_i(p)\theta_{mi}(p)[R_i(p)F_{mi}(p)+K_{ci}F_{ni}(p)]\left[\frac{D_i(p)}{K_i}Y_i+\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k-\sum_{k=1}^{n}\beta_{ik}(p)f_k\right]=$$

$$=\left[Y_{ref,i}\theta_{mi}(p)-Y_i\theta_{mi}(p)+\theta_{ni}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k\right]\mu_iK_{ci}F_{mi}(p)$$

or

$$\begin{aligned}&[Q_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)D_i(p)+\\&+K_{ci}D_i(p)Q_i(p)\theta_{mi}(p)F_{ni}(p)+\mu_iK_{ci}K_iF_{mi}(p)\theta_{mi}(p)]Y_i+\\&+K_iQ_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k+\\&+K_iK_{ci}F_{ni}(p)\theta_{mi}(p)Q_i(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k=\\&=K_i\mu_iK_{ci}F_{mi}(p)\theta_{mi}(p)Y_{ref}+F_{mi}(p)K_i\mu_iK_{ci}\theta_{ni}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k+\\&+K_iQ_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k+\\&+K_iK_{ci}Q_i(p)\theta_{mi}(p)F_{ni}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k.\end{aligned}\tag{5.4}$$

Taking $i=1, 2\ldots$, we obtain the complete set of equations for a combined control system with interrelated variables. In order to write these equations in matrix form, we put

$$\left.\begin{aligned}&Q_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)D_i(p)=a_{ii}(p),\\&D_i(p)F_{ni}(p)\theta_{mi}(p)Q_i(p)+\mu_iK_iF_{mi}(p)\theta_{mi}(p)=b_{ii}(p),\\&K_iQ_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)=c_{ii}(p),\\&K_iF_{ni}(p)Q_i(p)\theta_{mi}(p)=d_{ii}(p),\\&K_i\mu_iF_{mi}(p)\theta_{mi}(p)=l_{ii}(p),\\&F_{mi}(p)K_i\mu_i\theta_{ni}(p)=\rho_{ii}(p),\\&K_iQ_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)=N_{ii}(p).\end{aligned}\right\}\tag{5.5}$$

Making use of (5.5), we write

$$\begin{aligned}&[a_{ii}(p)+K_{ci}b_{ii}(p)]Y_i+c_{ii}(p)\alpha_{i1}(p)Y_1(p)+c_{ii}(p)\alpha_{i2}(p)Y_2(p)+\ldots\\&\ldots+c_{ii}(p)\alpha_{in}(p)Y_n(p)+d_{ii}(p)K_{ci}(p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=\\&=K_{ci}l_{ii}(p)Y_{iref}(p)+K_{ci}\rho_{ii}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k+N_{ii}(p)\sum_{k=1}^{n}\beta_{ik}(p)f_k+\\&+K_{ci}d_{ii}(p)\sum_{i=1}^{n}\beta_{ik}(p)f_k\qquad(i=1,\ 2,\ \ldots,\ n).\end{aligned}\tag{5.6}$$

In matrix form equations (5.6) are written as

$$A(p)Y(p)=B(p)Y_{ref}(p)+D(p)F(p).\tag{5.7}$$

For the sake of simplicity we henceforth omit the argument p, remembering that all equations are written in Laplace transforms.

In (5.7)

$$A(p)=\begin{Vmatrix} a_{11}+K_{c1}b_{11} & (c_{11}+K_{c1}d_{11})\alpha_{12} & \dots & (c_{11}+K_{c1}d_{11})\alpha_{1n} \\ (c_{22}+K_{c2}d_{22})\alpha_{21} & a_{22}+K_{c2}b_{22} & \dots & (c_{22}+K_{c2}d_{23})\alpha_{2n} \\ \dots & \dots & \dots & \dots \\ (c_{il}+K_{cl}d_{il})\alpha_{l1} & \dots & \dots & (c_{il}+K_{cl}d_{il})\alpha_{in} \\ \dots & \dots & \dots & \dots \\ (c_{nn}+K_{cn}d_{nn})\alpha_{n1} & (c_{nn}+K_{cn}d_{nn})\alpha_{n2} & \dots & a_{nn}+K_{cn}b_{nn} \end{Vmatrix}, \tag{5.8}$$

$$Y(p)=\begin{Vmatrix} Y_1 \\ Y_2 \\ \dots \\ Y_l \\ \dots \\ Y_n \end{Vmatrix}, \quad Y_{ref}(p)=\begin{Vmatrix} Y_{ref1} \\ Y_{ref2} \\ \dots \\ Y_{refl} \\ \dots \\ Y_{refn} \end{Vmatrix}, \quad F(p)=\begin{Vmatrix} f_1 \\ f_2 \\ \dots \\ f_l \\ \dots \\ f_n \end{Vmatrix},$$

$$B=\begin{Vmatrix} K_{c1}l_{11} & 0 & \dots & 0 \\ 0 & K_{c2}l_{11} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & K_{cn}l_{nn} \end{Vmatrix},$$

$$D=\begin{Vmatrix} [K_{c1}(\rho_{11}+d_{11})+N_{11}]\beta_{11} & [K_{c1}(\rho_{11}+d_{11})+N_{11}]\beta_{12}\dots[K_{c1}(\rho_{11}+d_{11})+N_{11}]\beta_{1n} \\ [K_{c2}(\rho_{22}+d_{22})+N_{22}]\beta_{21} & [K_{c2}(\rho_{22}+d_{22})+N_{22}]\beta_{22}\dots[K_{c2}(\rho_{22}+d_{22})+N_{22}]\beta_{2n} \\ \dots & \dots \\ [K_{cj}(\rho_{jj}+d_{jj})+N_{jj}]\beta_{j1} & [K_{cj}(\rho_{jj}+d_{jj})+N_{jj}]\beta_{j2}\dots[K_{cj}(\rho_{jj}+d_{jj})+N_{jj}]\beta_{jn} \\ \dots & \dots \\ [K_{cn}(\rho_{nn}+d_{nn})+N_{nn}]\beta_{n1} & [K_{cn}(\rho_{nn}+d_{nn})+N_{nn}]\beta_{n2}\dots[K_{cn}(\rho_{nn}+d_{nn})+N_{nn}]\beta_{nn} \end{Vmatrix}.$$

Here $\beta_{ii}=1$.

From (5.7) we obtain an expression for the Laplace-transform matrix $Y(p)$ of the controlled variables:

$$Y(p)=A^{-1}(p)[B(p)Y_{ref}(p)+D(p)F(p)]. \tag{5.9}$$

The matrix (5.9) should be represented in explicit form before explicit expressions for each controlled variable can be written. First we find the inverse $A^{-1}(p)$. The transpose A_t is given by

$$A_t(p)=\begin{Vmatrix} a_{11}+K_{c1}b_{11} & (c_{22}+K_{c2}d_{22})\alpha_{21} & \dots & (c_{nn}+K_{cn}d_{nn})\alpha_{n1} \\ (c_{11}+K_{c1}d_{11})\alpha_{12} & a_{22}+K_{c2}b_{22} & \dots & (c_{nn}+K_{cn}d_{nn})\alpha_{n2} \\ \dots & \dots & \dots & \dots \\ (c_{11}+K_{c1}d_{11})\alpha_{1i} & \dots & \dots & (c_{nn}+K_{cn}d_{nn})\alpha_{ni} \\ \dots & \dots & \dots & \dots \\ (c_{11}+K_{c1}d_{11})\alpha_{1n} & (c_{22}+K_{c2}d_{22})\alpha_{2n} & \dots & a_{nn}+K_{cn}b_{nn} \end{Vmatrix}. \tag{5.10}$$

From the transpose we obtain the inverse:

$$A^{-1}(p)=\frac{1}{\Delta}\begin{Vmatrix} A_{11}(p) & (-1)^{1+2}A_{12}(p) & \dots & (-1)^{1+n}A_{1n}(p) \\ -A_{21}(p) & A_{22}(p) & \dots & (-1)^{2+n}A_{2n}(p) \\ \dots & \dots & \dots & \dots \\ (-1)^{j+1}A_{j1}(p) & (-1)^{j+2}A_{j2}(p) & \dots & (-1)^{j+n}A_{jn}(p) \\ \dots & \dots & \dots & \dots \\ (-1)^{n+1}A_{n1}(p) & (-1)^{n+2}A_{n2}(p) & \dots & (-1)^{n+n}A_{nn}(p) \end{Vmatrix}, \tag{5.11}$$

where $(-1)^{j+i}A_{ji}(p)$ is the cofactor of the element at the intersection of the j-th row and the i-th column in $A_t(p)$, Δ is the determinant of the matrix $A(p)$ (it is clearly implied that $A(p)$ is a regular matrix). Multiplying out the matrices in brackets in (5.9), we find

$$B(p)Y(p)=\begin{Vmatrix} K_{c1}l_{11}Y_{ref1} \\ K_{c2}l_{22}Y_{ref2} \\ \dots \\ K_{cj}l_{jj}Y_{refj} \\ \dots \\ K_{cn}l_{nn}Y_{refn} \end{Vmatrix}, \tag{5.12}$$

$$D(p)F(p)=\begin{Vmatrix} [K_{c1}(\rho_{11}+d_{11})+N_{11}]\sum_{k=1}^{n}\beta_{1k}f_k \\ [K_{c2}(\rho_{22}+d_{22})+N_{22}]\sum_{k=1}^{n}\beta_{2k}f_k \\ \cdots\cdots\cdots\cdots \\ [K_{cj}(\rho_{jj}+d_{jj})+N_{jj}]\sum_{k=1}^{n}\beta_{jk}f_k \\ \cdots\cdots\cdots\cdots \\ [K_{cn}(\rho_{nn}+d_{nn})+N_{nn}]\sum_{k=1}^{n}\beta_{nk}f_k \end{Vmatrix}. \tag{5.13}$$

Inserting for $A^{-1}(p)$, $B(p)Y_{ref}(p)$, and $D(p)F(p)$ in equation (5.9) their expressions from (5.11), (5.12), and (5.13) and performing matrix multiplication, we obtain after simple manipulations an explicit expression for the matrix of controlled variables:

$$Y(p)=\frac{1}{\Delta}\begin{Vmatrix} \sum_{k=1}^{n}(-1)^{1+k}A_{1k}K_{ck}l_{kk}Y_{k\,ref} \\ \sum_{k=1}^{n}(-1)^{2+k}A_{2k}K_{ck}l_{kk}Y_{k\,ref} \\ \cdots\cdots\cdots\cdots \\ \sum_{k=1}^{n}(-1)^{j+k}A_{jk}K_{ck}l_{kk}Y_{k\,ref} \\ \cdots\cdots\cdots\cdots \\ \sum_{k=1}^{n}(-1)^{n+k}A_{nk}K_{ck}l_{kk}Y_{k\,ref} \end{Vmatrix}+$$

$$+\frac{1}{\Delta}\begin{Vmatrix} \sum_{i=1}^{n}(-1)^{1+i}A_{1i}[K_{ci}(\rho_{ii}+d_{ii})+N_{ii}]\sum_{k=1}^{n}\beta_{1k}f_k \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n}(-1)^{j+i}A_{ji}[K_{ci}(\rho_{ii}+d_{ii})+N_{ii}]\sum_{k=1}^{n}\beta_{1k}f_k \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n}(-1)^{n+1}A_{ni}[K_{ci}(\rho_{ii}+d_{ii})+N_{ii}]\sum_{k=1}^{n}\beta_{1k}f_k \end{Vmatrix}. \tag{5.14}$$

The expression for any controlled variable is obtained from (5.14) by equating the corresponding elements of the columns in these matrices. The expression for any j-th variable is thus written as

$$Y_j(p)=\frac{1}{\Delta}\sum_{k=1}^{n}(-1)^{j+k}A_{jk}K_{ck}l_{kk}Y_{k\,ref}(p)+$$

$$+\frac{1}{\Delta}\sum_{i=1}^{n}(-1)^{j+i}A_{ji}[K_{ci}(\rho_{ii}+d_{ii})+N_{ii}]\sum_{k=1}^{n}\beta_{ik}f_k. \tag{5.15}$$

The transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input. In single-variable systems the transfer function is the ratio of the Laplace transform of the controlled variable to the Laplace transform of the reference value, the load being ignored. We see from (5.15) that even if the component dependent on load (or disturbance) f_k (the second term in the right-hand side) is neglected, the output $Y_i(p)$ depends on all $Y_{i\,ref}(p)$. The concepts of a transfer function and a generalized transfer function will be very useful in this case.

The transfer function for the j-th controlled variable is defined as the ratio of Laplace transforms of the j-th controlled variable to the j-th reference value, disturbances being ignored.

If $f_i = 0$, we have from (5.15)

$$\frac{Y_j(p)}{Y_{j\,\mathrm{ref}}(p)} = \frac{A_{jj}(p)\,K_{c\,j}l_{jj}(p)}{\Delta} + \frac{1}{Y_{j\,\mathrm{ref}}(p)\,\Delta}\sum_{\substack{k=1\\k\neq j}}^{n}(-1)^{j+k}K_{c\,k}A_{jk}l_{kk}Y_{k\,\mathrm{ref}}(p). \tag{5.16}$$

Defining the generalized transfer function along the same lines as in single-variable control systems, we obtain

$$\frac{Y_j(p)}{Y_{j\,\mathrm{ref}}(p)} = \frac{A_{jj}(p)\,K_{c\,j}l_{jj}(p)}{\Delta} + \frac{1}{\Delta}\sum_{\substack{k=1\\k\neq j}}^{n}(-1)^{j+k}K_{c\,k}A_{jk}l_{kk}(p)\,\frac{Y_{k\,\mathrm{ref}}(p)}{Y_{j\,\mathrm{ref}}(p)} +$$

$$+\frac{1}{\Delta}\sum_{i=1}^{n}(-1)^{j+i}A_{ji}(p)\,[K_{c\,i}(\rho_{ii}(p)+d_{ii}(p))+N_{ii}(p)]\,\frac{1}{Y_{j\,\mathrm{ref}}(p)}\sum_{k=1}^{n}\beta_{ik}f_k(p). \tag{5.17}$$

The physical content of these expressions for transfer functions is quite obvious. The first term in either expression is the ordinary transfer function of a single-variable system; the second term in (5.16) and (5.17) gives the contribution to transfer function from the coupling of the given variable to other variables; finally, the third term in (5.17) shows to what extent the transfer function is influenced by self-load and by load or disturbance in other controlled variables.

Combined multivariable control systems considered in this chapter are conveniently analyzed with the aid of the generalized transfer function. The characteristic equation of the entire multivariable system has the form

$$\Delta = 0. \tag{5.18}$$

In what follows we consider some quality aspects of combined multivariable systems.

§ 5.3. STEADY-STATE OPERATION

The state of rest is a particular case of steady-state operation. The statics equations for this case can be derived from the theorem of limiting values. A statics equation is obtained from (5.15) by putting $p = 0$.

We consider two different cases:

(a) the case of proportional subsystems, and

(b) the case of integral subsystems (both in relation to the self-load).

(a) PROPORTIONAL SUBSYSTEMS

Using the nomenclature of (5.5) and putting $p = 0$, we write

$$\left.\begin{aligned} &a_{ii}(0) = Q_i(0)\,\theta_{mi}(0)\,R_i(0)\,F_{mi}(0)\,D_i(0) = 1\theta_{mi}(0),\\ &b_{ii}(0) = \mu_i K_i \theta_{mi}(0), \quad c_{ii}(0) = K_i\theta_{ni}(0),\\ &d_{ii}(0) = 0, \qquad l_{ii} = \mu_i K_i \theta_{mi}(0),\\ &\delta_{ii}(0) = \theta_{ni}(0)\,\mu_i K_i, \qquad N_{ii}(0) = K_i\theta_{mi}(0). \end{aligned}\right\} \tag{5.19}$$

The steady-state equation for the j-th controlled variable of a proportional system has the form

$$Y_j(0)=\frac{1}{\Delta(0)}\sum_{k=1}^{n}(-1)^{j+k}A_{jk}(0)K_{ck}l_{kk}(0)Y_{\text{ref }k}(0)+$$
$$+\frac{1}{\Delta(0)}\sum_{i=1}^{n}(-1)^{j+i}A_{ji}(0)[K_{ci}(\rho_{ii}(0)+d_{ii}(0))+N_{ii}(0)]\sum_{k=1}^{n}\beta_{ik}(0)f_k(0)$$

or, making use of (5.19)

$$Y_j(0)=\frac{1}{\Delta(0)}\sum_{k=1}^{n}(-1)^{j+k}A_{jk}(0)K_{ck}\mu_kK_k\theta_{mk}(0)Y_{\text{ref}}(0)+$$
$$+\frac{1}{\Delta(0)}\sum_{i=1}^{n}(-1)^{j+i}A_{ji}(0)[\mu_iK_iK_{ci}\theta_{ni}(0)+K_i\theta_{mi}(0)]\sum_{k=1}^{n}\beta_{ik}(0)f_k. \qquad (5.20)$$

Here

$$\Delta(0)=\begin{vmatrix}\theta_{m1}(0)+ & K_1\theta_{n1}(0)\alpha_{12} & \dots & K_1\theta_{n1}(0)\alpha_{1n}\\ +K_{c1}(1+\mu_1K_{c1}K_i\theta_{mi}(0)) & & & \\ K_2\theta_{n2}(0)\alpha_{21} & \theta_{m2}(0)+ & \dots & K_2\theta_{n2}(0)\alpha_{2n}\\ & +K_{c2}(1+\mu_2K_2\theta_{m2}(0)) & & \\ \dots\dots\dots\dots & \dots\dots\dots\dots & & \dots\dots\dots\dots\\ K_j\theta_{nj}(0)\alpha_{j1} & \theta_{mj}(0)+ & \dots & K_j\theta_{nj}(0)\alpha_{jn}\\ & +K_{cj}(1+\mu_jK_{cj}K_j\theta_{mi}(0)) & & \\ \dots\dots\dots\dots & \dots\dots\dots\dots & & \dots\dots\dots\dots\\ K_n\theta_{nn}(0)\alpha_{n1} & \dots & \dots & \theta_{mn}(0)+\\ & & & +K_{cn}\times\\ & & & \times(1+\mu_nK_{cn}K_n\theta_{mn}(0))\end{vmatrix} \qquad (5.21)$$

From (5.20) we can find the steady-state value of the j-th variable for given loads f_k, if all the system parameters are known.

Before proceeding to determine the properties of an n-variable system under static conditions, we shall consider the application of the above equations to a three-variable system. From (5.15) with $n=3$ we have for, say, the 1st controlled variable

$$Y_1=\frac{1}{\Delta_3}\sum_{k=1}^{3}(-1)^{1+k}A_{1k}K_{ck}l_{kk}Y_{k\,\text{ref}}+$$
$$+\frac{1}{\Delta}\sum_{i=1}^{3}(-1)^{1+i}A_{1i}[K_{ci}(\rho_{ii}+d_{ii})+N_{ii}]\sum_{k=1}^{3}\beta_{ik}f_k=$$
$$=\frac{1}{\Delta_3}\{A_{11}K_{c1}l_{11}Y_{\text{ref}\,1}-A_{12}K_{c2}l_{22}Y_{\text{ref}\,2}+A_{13}K_{c3}l_{33}Y_{\text{ref}\,3}+$$
$$+A_{11}[K_{c1}(\rho_{11}+d_{11})+N_{11}][\beta_{11}f_1+\beta_{12}f_2+\beta_{13}f_3]-$$
$$-A_{12}[K_{c2}(\rho_{22}+d_{22})+N_{22}][\beta_{21}f_1+\beta_{22}f_2+\beta_{23}f_3]+$$
$$+A_{13}[K_{c3}(\rho_{33}+d_{33})+N_{33}][\beta_{31}f_1+\beta_{32}f_2+\beta_{33}f_3]\}. \qquad (5.22)$$

Here

$$\Delta_3=\begin{vmatrix}a_{11}+K_{c1}b_{11} & (c_{11}+K_{c1}d_{11})\alpha_{12} & (c_{11}+K_{c1}d_{11})\alpha_{13}\\ (c_{22}+K_{c2}d_{22})\alpha_{21} & a_{22}+K_{c2}b_{22} & (c_{22}+K_{c2}d_{22})\alpha_{23}\\ (c_{33}+K_{c3}d_{33})\alpha_{31} & (c_{33}+K_{c3}d_{33})\alpha_{32} & a_{33}+K_{c3}b_{33}\end{vmatrix}. \qquad (5.22')$$

The transpose in this case is

$$A_{f3}=\begin{Vmatrix} a_{11}+K_{c1}b_{11} & (c_{22}+K_{c2}d_{22})\,\alpha_{21} & (c_{33}+K_{c3}d_{33})\,\alpha_{31} \\ (c_{11}+K_{c1}d_{11})\,\alpha_{12} & a_{22}+K_{c2}b_{22} & (c_{33}+K_{c3}d_{33})\,\alpha_{32} \\ (c_{11}+K_{c1}d_{11})\,\alpha_{13} & (c_{22}+K_{c2}d_{22})\,\alpha_{23} & a_{33}+K_{c3}b_{33} \end{Vmatrix},$$

whence

$$A_{11}=\begin{vmatrix} a_{22}+K_{c2}b_{22} & (c_{33}+K_{c3}d_{33})\,\alpha_{32} \\ (c_{22}+K_{c2}d_{22})\,\alpha_{23} & a_{33}+K_{c3}b_{33} \end{vmatrix},$$

$$A_{12}=\begin{vmatrix} (c_{11}+K_{c1}d_{11})\,\alpha_{12} & (c_{33}+K_{c3}d_{33})\,\alpha_{32} \\ (c_{11}+K_{c1}d_{11})\,\alpha_{13} & a_{33}+K_{c3}b_{33} \end{vmatrix},$$

$$A_{13}=\begin{vmatrix} (c_{11}+K_{c1}d_{11})\,\alpha_{12} & a_{22}+K_{c2}b_{22} \\ (c_{11}+K_{c1}d_{11})\,\alpha_{13} & (c_{22}+K_{c2}d_{22})\,\alpha_{23} \end{vmatrix}.$$

Under steady-state conditions, we have from (5.19)

$$\Delta_3(0)=\begin{vmatrix} \theta_{m1}(0)+ & K_1\theta_{m1}(0)\,\alpha_{12} & K_1\theta_{m1}(0)\,\alpha_{13} \\ +K_{c1}\mu_1K_1\theta_{m1}(0) & & \\ K_2\theta_{m2}(0)\,\alpha_{21} & \theta_{m2}(0)+ & K_2\theta_{m2}(0)\,\alpha_{23} \\ & +K_{c2}\mu_2K_2\theta_{m2}(0) & \\ K_3\theta_{m3}(0)\,\alpha_{31} & K_3\theta_{m3}(0)\,\alpha_{32} & \theta_{m3}(0)+ \\ & & +K_{c3}\mu_3K_3\theta_{m3}(0) \end{vmatrix},$$

$$A_{11}(0)=\begin{vmatrix} \theta_{m2}(0)+K_{c2}\mu_2K_2\theta_{m2}(0) & K_3\theta_{m3}(0)\,\alpha_{32} \\ K_2\theta_{m2}(0)\,\alpha_{23} & \theta_{m3}(0)+K_{c3}\mu_3k_3\theta_{m3}(0) \end{vmatrix},$$

$$A_{12}(0)=\begin{vmatrix} K_1\theta_{m1}(0)\,\alpha_{12} & K_3\theta_{m3}(0)\,\alpha_{32} \\ K_1\theta_{m1}(0)\,\alpha_{13} & \theta_{m3}(0)+K_{c3}\mu_3K_3\theta_{m3}(0) \end{vmatrix},$$

$$A_{13}(0)=\begin{vmatrix} K_1\theta_{m1}(0)\,\alpha_{12} & \theta_{m2}(0)+K_{c2}\mu_2K_2\theta_{m2}(0) \\ K_1\theta_{m1}(0)\,\alpha_{13} & K_2\theta_{m2}(0)\,\alpha_{23} \end{vmatrix}$$

or substituting the steady-state expressions in (5.22), we find after simple manipulations

$$\begin{aligned} Y_1(0)=\{&K_{c1}\mu_1K_1\theta_{m1}(0)\,[(\theta_{m2}(0)+K_{c2}\mu_2K_2\theta_{m2}(0))\,(\theta_{m3}(0)+ \\ &+K_{c3}\mu_3K_3\theta_{m3}(0)-K_2K_3\theta_{n2}(0)\,\theta_{n3}(0)\,\alpha_{23}\alpha_{32})]\,Y_{\text{ref}1}- \\ &-K_{c2}\mu_2K_2\theta_{m2}\,[K_1\theta_{m1}(0)\,\alpha_{12}\,(\theta_{m3}(0)+K_{c3}\mu_3K_3\theta_{m3}(0))- \\ &-K_1\theta_{n1}(0)\,K_3\theta_{n3}(0)\,\alpha_{13}\alpha_{32}]\,Y_{\text{ref}2}+K_{c3}\mu_3K_3\theta_{m3}\,[K_1\theta_{m1}(0)\,K_2\theta_{n2}(0)\,\alpha_{12}\alpha_{13}- \\ &-K_1\theta_{n1}\alpha_{13}\,(\theta_{m2}(0)+K_{c2}\mu_2K_2\theta_{m2}(0))]\,Y_{\text{ref}3}+[(\theta_{m2}(0)+ \\ &+K_{c2}\mu_2K_2\theta_{m2}(0))\,(\theta_{m3}(0)+K_{c3}\mu_3K_3\theta_{m3}(0))- \\ &-K_2K_3\theta_{m2}(0)\,\theta_{m3}(0)\,\alpha_{23}\alpha_{32}]\,[K_{c1}\mu_1K_1\theta_{m1}(0)+K_1\theta_{m1})\,(f_1+\beta_{12}f_2+ \\ &+\beta_{13}f_3)]-[K_1\theta_{n1}(0)\,(\theta_{m3}+K_{c3}\mu_3K_3K_3\theta_{m3}(0))- \\ &-K_1\theta_{n1}(0)\,K_3\theta_{n3}(0)\,\alpha_{13}\alpha_{32}]\,(K_{c2}K_2\mu_2\theta_{n2}(0)+K_2\theta_{m2})\,(\beta_{21}f_1+f_2+ \\ &+\beta_{23}f_3)+[(K_1\theta_{n1}(0)\,K_2\theta_{n2}(0)\,\alpha_{12}\alpha_{23}-[\theta_{m2}(0)+ \\ &+K_{c2}\mu_2K_2\theta_{m2}(0)\,K_1\theta_{n1}(0)\,\alpha_{13}]\,(K_{c3}K_3\mu_3\theta_{n3}(0)+K_3\theta_{m3}(0))\,(\beta_{31}f_1+ \\ &+\beta_{32}f_2+f_3)\}\{[\theta_{m1}(0)+K_{c1}\mu_1K_1\theta_{m1}(0)]\,[\theta_{m2}(0)+ \\ &+K_{c2}\mu_2K_2\theta_{m2}(0)]\,[\theta_{m3}(0)+K_{c3}\mu_3K_3\theta_3(0)]+ \\ &+K_1\theta_{m1}(0)\,K_2\theta_{m2}(0)\,K_3\theta_{m3}(0)\,[\alpha_{12}\alpha_{23}\alpha_{31}+\alpha_{21}\alpha_{32}\alpha_{13}]- \\ &-K_3K_1\theta_{m3}(0)\,\theta_{m1}(0)\,\alpha_{13}\alpha_{31}\,[\theta_{m2}(0)+K_{c2}\mu_2K_2\theta_{m2}(0)]- \\ &-K_2K_3\theta_{m2}(0)\,\theta_{m3}(0)\,\alpha_{32}\alpha_{23}\,[\theta_{m1}(0)+K_{c1}\mu_1K_1\theta_{m1}(0)]- \\ &-K_1K_2\theta_{m1}(0)\,\theta_{m2}(0)\,\alpha_{12}\alpha_{21}\}^{-1}. \end{aligned} \tag{5.23}$$

We are now in a position to draw some conclusions from this example of a three-variable system that can be readily generalized to n-variable systems. At the outset we have assumed that the structure (i.e., the stabilizer $\frac{F_{ni}(p)}{F_{mi}(p)}$ and the point of its connection to the system) permits

indefinitely increasing the gain in any of the subsystems without loss of stability. Let the gain parameters of all the three subsystems K_{ci}, $i = 1, 2, 3$, increase indefinitely. Then, as it follows from (5.23),

$$\lim_{K_{ci}\to\infty} Y_1(0) = Y_{ref\,1} + \frac{\theta_{n1}(0)}{\theta_{m1}(0)}[f_1 + \beta_{12}f_2 + \beta_{13}f_3] \quad (i = 1, 2, 3). \tag{5.24}$$

In other words, indefinite increase in the subsystem gains under steady-state conditions makes the output equal to the reference value Y_{ref} appropriately modified by the various disturbances. The effect of disturbances depends on load coupling coefficients $\beta_{ik}(0)$ and the coefficient $\frac{\theta_{ni}(0)}{\theta_{mi}(0)}$. In the particular case $\theta_{ni}(0) = 0$, which can be implemented without any difficulty, we have

$$\lim Y_1(0) = Y_{1\,ref}. \tag{5.25}$$

This result is obtained for the steady-state conditions, since we have taken $F_{ni}(0) = 0$. This is a natural assumption because in this case, as has been shown in /39/, increase in gain improves the accuracy.

Two particular cases deserve special attention: one is the case of stabilization by proportional feedback and the other the case of a mixed-type stabilizer. Expression (5.24) clearly does not apply in these cases, and we will have to consider them separately. The following general conclusions thus follow from the statics of combined multivariable systems with proportional subsystems:

1. Increase in subsystem gains leads to decoupling, eliminating all interrelationships between the controlled variables under steady-state conditions.

2. Increase in gain improves the accuracy of each controlled variable, and if $\theta_{ni}(0) = 0$, all disturbances are rejected.

If the gain factors are finite, these conclusions are true only to a certain degree. In the case of finite, but sufficiently large gain, we can speak of decoupling or disturbance rejection under steady-state conditions to an accuracy of ε only. In the general case, the actual output values for each load and for each set of gain parameters can be obtained from (5.23).

We see from (5.23) that each controlled variable depends not only on the disturbances and its own reference value but also on the reference values of all the other controlled variables.

Our conclusions are based on the particular case of a three-variable control system. Generalization to n-variable systems obtains from the following considerations. It is clear from equation (5.22) and from the construction of Δ and A_{ij}, that the highest degree of K_c, equal to the highest degree of K_c in the expansion of Δ, occurs only in that term of the numerator which corresponds to the reference value of the variable itself. This explains why structures of this class are inherently capable of suppressing the effect of other extraneous components.

(b) INTEGRAL SUBSYSTEMS

A system is integral if and only if an integrating element is included in the corresponding single-loop configuration; the integrating element

should be unstabilized and must not constitute a structural component of the plant /39/. Under these conditions we have for the steady-state case

$$\left.\begin{array}{lll} a_{ii}(0)=0, & b_{ii}(0)=\mu_i K_i \theta_{mi}(0), & c_{ii}(0)=0, \\ d_{ii}(0)=0, & l_{ii}(0)=\mu_i K_i \theta_{mi}(0), & \rho_{ii}(0)=K_i \mu_i \theta_{ni}(0), \\ & N_{ii}(0)=0. & \end{array}\right\} \tag{5.26}$$

Substituting (5.26) in (5.22), we find

$$Y_1=\frac{1}{\Delta_3}\sum_{k=1}^{3}(-1)^{1+k}A_{1k}K_{ck}\mu_k K_k \theta_{mk}(0)+ \\ +\frac{1}{\Delta_3}\sum_{i=1}^{3}(-1)^{1+i}A_{1i}K_{ci}K_i\mu_1\theta_{n1}(0)\sum_{k=1}^{3}\beta_{ik}f_k. \tag{5.27}$$

Inserting for A_{ik} their expressions and making use of (5.26), we find

$$\left.\begin{array}{l} A_{11}=K_{c2}K_{c3}K_2K_3\mu_2\mu_3\theta_{m2}(0)\,\theta_{m3}(0), \\ A_{12}=0, \\ A_{13}=0, \end{array}\right\} \tag{5.28}$$

whence

$$Y_1=\frac{K_{c1}K_{c2}K_{c3}K_1K_2K_3\mu_1\mu_2\mu_3\theta_{m1}(0)\,\theta_{m2}(0)\,\theta_{m3}(0)}{K_{c1}K_{c2}K_{c3}\mu_1\mu_2\mu_3K_1K_2K_3\theta_{m1}(0)\,\theta_{m2}(0)\,\theta_{m3}(0)}Y_{ref1}+ \\ +\frac{K_{c1}K_{c2}K_{c3}K_1K_2K_3\mu_1\mu_2\mu_3\theta_{m2}(0)\,\theta_{m3}(0)\,\theta_{n1}(0)}{K_{c1}K_{c2}K_{c3}\mu_1\mu_2\mu_3K_1K_2K_3\theta_{m1}(0)\,\theta_{m2}(0)\,\theta_{m3}(0)}[f_1+\beta_{12}f_2+\beta_{13}f_3]= \\ =Y_{ref1}+\frac{\theta_{n1}(0)}{\theta_{m1}(0)}[f_1+\beta_{12}f_2+\beta_{13}f_3]. \tag{5.29}$$

In other words, in integral systems, without increasing the gain, we find that the steady-state output variable is equal to the corresponding reference value plus a contribution from all the loads. If we select $\theta_{ni}(0)=0$, the load contribution vanishes under steady-state conditions. In general, introduction of the factor $\frac{\theta_{n1}(0)}{\theta_{m1}(0)}$ makes the variable load dependent in integral systems also. In a number of cases this load dependence may prove to be quite profitable. It is actually utilized in the so-called compounding systems, e.g., an electric power station where proportional current feedback increases the voltage of the synchronous power generators when the load is increased.

By reversing the sign of $\frac{\theta_{n1}(0)}{\theta_{m1}(0)}$ the load can be made to increase or decrease the output value of integral systems in comparison with the reference value.

A significant feature of systems considered in this chapter is that load, or disturbance, is used as an additional factor for imparting certain desirable properties to the system as a whole and consequently to the individual controlled variables. It is clear from equations (5.29) and (5.24) that under steady-state conditions the output of both proportional (for $K_{ci}\to\infty$) and integral systems depends on the reference value and the properties of the transducer and all the loads. In proportional systems, in particular, the load can be employed to improve the accuracy, if the

gain is insufficiently high for meeting the accuracy standards. It will be clear from what follows that load can be utilized as an additional powerful factor for modifying the system dynamics.

§ 5.4. STABILITY

The dynamic properties of multivariable control systems are defined by equation (5.14). This equation corresponds to zero initial conditions. Introduction of nonzero initial conditions will not alter the structure of equation (5.14), only adding a matrix of initial conditions. There is an almost infinite variety of initial conditions, and no one particular set of conditions can be given preference. However, zero initial conditions have certain other advantages than a simple form of the equation. Analysis of system dynamics with zero initial conditions brings out those properties which are dependent solely on the system's structure and the numerical values of its parameters. This information is highly valuable, as it can be used as a foundation in the development of system design techniques. In what follows we therefore confine our investigation of system dynamics to cases with zero initial conditions.

The dynamic properties of any i-th controlled variable are specified by equation (5.15). This equation is used as a point of departure in our analysis. Let us first consider the stability of combined multivariable control systems. The stability of a multivariable system, like that of an ordinary linear system, is determined by the position of the roots of the characteristic equation. The characteristic equation is obtained by putting the system determinant Δ equal to zero, thus;

$$\Delta = 0,$$

or in expanded form

$$\Delta = \begin{vmatrix} a_{11} + K_{c1} b_1 & (c_{11} + K_{c1} d_{11}) \alpha_{12} & \dots & (c_{11} + K_{c1} d_{11}) \alpha_{1n} \\ (c_{22} + K_{c2} d_{22}) \alpha_{21} & a_{22} + K_{c2} b_{22} & \dots & (c_{22} + K_{c2} d_{22}) \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ (c_{jj} + K_{cj} d_{jj}) \alpha_{j1} & & a_{jj} + K_{cj} b_{jj} & (c_{jj} + K_{cj} d_{jj}) \alpha_{jn} \\ \dots & \dots & \dots & \dots \\ (c_{nn} + K_{cn} d_{nn}) \alpha_{n1} & \dots & \dots & a_{nn} + K_{cn} b_{nn} \end{vmatrix} = 0. \qquad (5.30)$$

At the outset let us note that the introduction of a transducer $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ (its input receives the overall load or disturbance) does not affect the stability of a combined control system as long as all $\theta_{mi}(p)$ have no right-half-plane zeros, i.e., if the transducers themselves are inherently stable. Indeed, it follows from the notation in (5.5) that each of the quantities a_{ii}, b_{ii}, c_{ii}, and d_{ii} contains the corresponding θ_{mi} as a factor, and θ_{mi} can therefore be taken outside the determinant from each row in (5.30); now if none of these θ_{mi}, $i = 1, 2, \dots, n$, has right-half-plane zeros, the stability of the entire system is independent of the transducer properties. This fundamental (though trivial) property leads to a very important structural corollary: if the structure of a combined multivariable control system (in the absence of load) remains stable at indefinitely high gain,

the combined control system generated by introducing external load or any other external disturbance into the original system through a transducer of a quite general kind also possesses infinite-gain stability. The only restriction in this case is the requirement of minimum transducer phase.

This proposition can be given a rigorous proof. Indeed, let the gain factors K_{ci} be related by the expression $K_i=\eta_i K$, as in Chapter Four. Expanding the determinant (5.30), we write the characteristic equation as

$$\prod_{i=1}^{n}\theta_{mi}(p)\left[F_N(p)+K_c F_{N1}(p)+K_c^2F_{N1}(p)+\right.$$
$$\left.+\ \dots\ +K_c^nF_{N,\,N-n}(p)\right]=0, \qquad (5.31)$$

or dividing through by K_c^n and putting $\frac{1}{K_c^i}=m^i$, we find

$$\prod_{i=1}^{n}\theta_{mi}(p)\left[m^nF_N(p)+m^{n-1}F_{N1}(p)+\ \dots\ +F_{N.\,N-n}(p)\right]=0. \qquad (5.32)$$

It follows from the results of a previous chapter that the difference in the degrees of the adjoining polynomials F_{Ni}, F_{Ni-1} is determined by the subsystem structures. If these structures are stable at infinite gain, the structure of the combined control system is also stable at infinite gain.

Introduction of load disturbance thus does not affect the stability of the system, so long as the transducer through which the load disturbances are fed complies with the requirement of minimum phase.

§ 5.5. DYNAMICS

The dynamic properties of multivariable systems, unlike their stability, depend not only on the poles but also on the zeros of the transfer function. The transfer function of ordinary multivariable systems is expressed by equation (5.16), and the generalized transfer function of combined control systems is represented by equation (5.17).

In order to elucidate the dynamic properties of structures (or, more precisely, the structural features of system dynamics), let us assume that the reference values have the form of unit step pulses (or that all the reference values vary according to the same relation, differing only in a scale factor). The factor $\frac{Y_{k\,\mathrm{ref}}(p)}{Y_{i\,\mathrm{ref}}(p)}$ can therefore be omitted, as it introduces only a scale correction. The factor $\frac{1}{Y_{i\,\mathrm{ref}}(p)}$ entering the second term in the right-hand side of (5.17), however, cannot be ignored. System dynamics are thus determined by the generalized transfer function (5.17).

Let us establish the dynamic properties of systems which remain stable as the gain K_{ci} is increased indefinitely, assuming fairly high gains from the start. From (5.17) we now have

$$\frac{Y_j(p)}{Y_{j\,\mathrm{ref}}(p)}=\frac{1}{\Delta}\left\{A_{jj}(p)K_{cj}l_{jj}(p)+\sum_{\substack{k=1\\k\neq j}}^{n}(-1)^{j+k}K_{ck}A_{jk}(p)\,l_{kk}(p)+\right.$$
$$\left.+\sum_{i=1}^{n}(-1)^{j+i}A_{ji}(p)\left[K_{ci}\left(\rho_{ii}(p)+d_{ii}(p)\right)+N_{ii}(p)\right]\frac{1}{Y_{j\,\mathrm{ref}}(p)}\sum_{k=1}^{n}\beta_{ik}f_k(p)\right\}. \qquad (5.33)$$

The structure of (5.33) in relation to a small parameter is found by expanding the numerator in the right-hand side of this equation in terms of K_c. Since A_{jk} ($k=1, \ldots, n$) are the cofactors of the corresponding elements in the transpose of the determinant (5.30) and since all the elements of (5.30) are linear combinations of K_c, the highest degree of K_c in the expanded cofactors A_{jk} is clearly $n-1$. Now, since each of the terms in the equation, with the exception of $N_{ii}(p)$ which multiply A_{ik}, is linear in K_{ci}, the highest degree of K_{ci} in the numerator of (5.33) is also n.

Let us now concentrate on the construction of the cofactors A_{ik}. From the construction of the system matrix and its transpose it is clear that only cofactors of the form A_{ii} can be expanded into expressions with components that are independent of the coupling coefficients α_{ik}. This follows from the fact that only cofactors of the form A_{ii} have diagonal elements corresponding to the diagonal elements of the original matrix. Keeping these remarks in mind, we write the transfer function (5.33) in the form

$$\begin{aligned} K(p) = \{ & m^n \varphi_{N_3}(p) + m^{n-1}\varphi_{N_2-1}(p) + \ldots + m\varphi_{N_2-(n-1)}(p) + \\ & + \varphi_{N_2-n}(p) + m^n \psi_{N_3}(p) f_1(\alpha_{ik}) + m^{n-1}\psi_{N_3-1}(p) f_2(\alpha_i k) + \ldots \\ & \ldots + \psi_{N_3-n}(p) f_n(\alpha_{ik}) + m^n \xi_{N_4}(p) f_1(\beta_{ik} f_k) + \ldots \\ & \ldots + \xi_{N_4-n}(p) f_n(\beta_{ik}) \Big\} \Big\{ \prod_{i=1}^{n} Q_{mi}(p) \left[m^n F_{N_1}(p) + m^{n-1} F_{N_1-1}(p) + \ldots \right. \\ & \left. \ldots + mF_{N_1-(n-1)}(p) + F_{N_1-n}(p) \right] \Big\}^{-1}, \end{aligned} \tag{5.34}$$

where $m = \frac{1}{K_c}$.

We see from (5.34) that the numerator in the right-hand side of (5.33) is a composite function of system parameters, gains, and loads. Let us try to elucidate, in as great detail as possible, the structure of polynomials in (5.34); this will enable us to reach some conclusions concerning the general structural properties of these systems.

First consider the denominator in (5.34). Since we work with structures which remain stable for $K_{ci} \to \infty$ or, equivalently, for $m \to 0$, the degrees of two adjoining polynomials differ at most by 2, i.e.,

$$N_{N_{1,\,l}} - N_{N_{1,\,l+1}} \leqslant 2.$$

Since A_{ii} is a triangular determinant constructed from elements of the same matrix as Δ, with the omission of one row and one column, the polynomials φ_{Ni} obey the same rule and for the degrees of two adjoining polynomials we have

$$\varphi_{N_{2l}} - \varphi_{N_{2l+1}} = N_{N_l} - N_{N_{l+1}}.$$

This conclusion is obviously also valid for the polynomials $\psi_{N_{3l}}$ and $\xi_{N_{4l}}$.

Let us now establish a relationship between the absolute value of the degrees of the polynomials in the numerator and the denominator. The highest degree in Δ is greater than the degree of A_{ii} by an amount equal to the degree of the term $a_{ii} + K_c b_{ii}$. From (5.5) it is clear that the degree of $a_{ii}(p)$ is greater than the degree of $b_{ii}(p)$, so that the highest degree in Δ is greater than the degree of $A_{ii}(p)$ by an amount equal to the degree of $a_{ii}(p)$. Making use of the expression for $b_{ii}(p)$ we conclude that the highest

degree in Δ, or equivalently the degree of F_{N0}, is greater than the degree of φ_{N_i}, the difference being "degree a_{ii}–degree $F_{mi}(p)\theta_{mi}(p)$." If the structure has infinite-gain stability and if the additional conditions are also satisfied, the system dynamics is determined by the degenerate equation, which has the general form

$$K_{\deg}(p)=\frac{\varphi_{N_2-n}(p)+\psi_{N_3-n}(p)f_n(\alpha_{ik})+\xi_{N_4-n}(p)f_n(\beta_{ik}f_k)}{\prod_{i=1}^{n}\theta_{mi}(p)F_{N_i-n}(p)}. \tag{5.35}$$

Let us now find explicit expressions for F_{N_1-n}, φ_{N_2-n}, ψ_{N_3-n}, and ξ_{N_4-n}. This will provide us with a starting point for the reconstruction of the transients and for the determination of the fundamental properties of systems with sufficiently high gain.

We will make the calculations for the particular case of three interrelated variables, and then generalize to the case of n interrelated controlled variables.

The equation for the 1st controlled variable is obtained from (5.17), where we put $n=3$:

$$\begin{aligned}\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)}=&\frac{A_{11}(p)K_{c1}l_{11}(p)}{\Delta}-\frac{1}{\Delta}K_{c2}A_{12}(p)l_{22}(p)\frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}+\\&+\frac{1}{\Delta}K_{c3}A_{13}(p)l_{33}(p)\frac{Y_{3\,\mathrm{ref}}(p)}{Y_{2\,\mathrm{ref}}(p)}+\\&+\frac{1}{\Delta}A_{11}[K_{c1}(\rho_{11}(p)+d_{11}(p))+N_{11}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}(f_1(p)+\beta_{12}f_2(p)+\\&+\beta_{13}f_3(p))-\frac{1}{\Delta}A_{12}[K_{c2}(\rho_{22}(p)+d_{22}(p))+N_{22}(p)]\times\\&\times\frac{1}{Y_{1\,\mathrm{ref}}(p)}(\beta_{21}f_1(p)+f_2+p_{23}f_3(p))+\frac{1}{\Delta}A_{13}[K_{c3}(p_{33}(p)+\\&+d_{33}(p))+N_{33}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}(\beta_{31}f_1(p)+\beta_{32}f_2(p)+f_3(p)),\end{aligned} \tag{5.36}$$

where $\beta_{ii}=1$ and

$$\Delta=\begin{vmatrix}a_{11}(p)+K_{c1}b_{11}(p) & [c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{12} & [c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{13}\\ [c_{22}(p)+K_{c2}d_{22}(p)]\alpha_{21} & a_{22}(p)+K_{c2}b_{22}(p) & [c_{22}(p)+K_{c2}d_{22}(p)]\alpha_{23}\\ [c_{33}(p)+K_{c3}d_{33}(p)]\alpha_{31} & [c_{33}(p)+K_{c3}d_{33}(p)]\alpha_{32} & a_{33}(p)+K_{c3}b_{33}(p)\end{vmatrix}, \tag{5.37}$$

$$A_{11}=\begin{vmatrix}a_{22}(p)+K_{c2}b_{22}(p) & [c_{33}(p)+K_{c3}d_{33}(p)]\alpha_{32}\\ [c_{22}(p)+K_{c2}d_{22}(p)]\alpha_{23} & a_{33}(p)+K_{c3}b_{33}(p)\end{vmatrix}, \tag{5.38}$$

$$A_{12}=\begin{vmatrix}[c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{12} & [c_{33}(p)+K_{c3}d_{33}(p)]\alpha_{32}\\ [c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{13} & a_{33}(p)+K_{c3}b_{33}(p)\end{vmatrix}, \tag{5.39}$$

$$A_{13}=\begin{vmatrix}[c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{12} & a_{22}(p)+K_{c2}b_{22}(p)\\ [c_{11}(p)+K_{c1}d_{11}(p)]\alpha_{13} & [c_{22}(p)+K_{c2}d_{22}(p)]\alpha_{23}\end{vmatrix}. \tag{5.40}$$

Let the various gains K_{ci} be of the same order of magnitude, so that we may put $K_{c1}=K_{c2}=K_{c3}=K_c$. This is not a fundamental restriction, since we can always make use of the relation $K_i=\eta_iK$. Dividing the numerator and the denominator in (5.36) by K_c^3 and taking $K_c\to\infty$, we obtain after simple manipulations a degenerate equation in the form

$$\begin{aligned}\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)}=&\frac{1}{\Delta_{\deg 3}}\Big[A_{11b}(p)\Big\{l_{11}(p)+[\rho_{11}(p)+d_{11}(p)]\times\\&\times\frac{1}{Y_{1\,\mathrm{ref}}(p)}[f_1(p)+\beta_{12}f_2(p)+\beta_{13}f_3(p)]\Big\}-A_{12b}(p)\Big\{l_{22}(p)\frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}+\\&+[\rho_{22}(p)+d_{22}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[\beta_{21}f_1(p)+f_2(p)+\beta_{23}(p)]\Big\}+\\&+A_{13b}(p)\Big\{l_{33}(p)\frac{Y_{3\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}+[\rho_{33}(p)+d_{33}(p)]\times\frac{1}{Y_{1\,\mathrm{ref}}(p)}[\beta_{31}f_1(p)+\beta_{32}f_2(p)+f_3(p)]\Big\}\Big],\end{aligned} \tag{5.41}$$

where

$$\Delta_{\deg 3}=\begin{vmatrix} b_{11}(p) & d_{11}(p)\alpha_{12} & d_{11}(p)\alpha_{13} \\ d_{22}(p)\alpha_{21} & b_{22}(p) & d_{22}(p)\alpha_{23} \\ d_{33}(p)\alpha_{31} & d_{33}(p)\alpha_{32} & b_{33}(p) \end{vmatrix}, \tag{5.42}$$

$$A_{11b}=\begin{vmatrix} b_{22}(p) & d_{33}(p)\alpha_{32} \\ d_{22}(p)\alpha_{23} & b_{33}(p) \end{vmatrix}, \tag{5.43}$$

$$A_{12b}=\begin{vmatrix} d_{11}(p)\alpha_{12} & d_{33}(p)\alpha_{32} \\ d_{11}(p)\alpha_{13} & b_{33}(p) \end{vmatrix}, \tag{5.44}$$

$$A_{13b}=\begin{vmatrix} d_{11}(p)\alpha_{12} & b_{22}(p) \\ d_{11}(p)\alpha_{13} & d_{22}(p)\alpha_{23} \end{vmatrix}. \tag{5.45}$$

The degenerate equation of the 1st controlled variable in an n-variable system obviously has the form

$$\frac{Y_j(p)}{Y_{j\,\mathrm{ref}}(p)}=\frac{1}{\Delta_b}\sum_{k=1}^{n}A_{jkb}(p)(-1)^{j+k}\times$$
$$\times\left\{l_{kk}(p)\frac{Y_{k\,\mathrm{ref}}(p)}{Y_{j\,\mathrm{ref}}(p)}+[\rho_{kk}(p)+d_{kk}(p)]\frac{1}{Y_{j\,\mathrm{ref}}(p)}\sum_{i=1}^{n}\beta_{ki}f_i\right\}.$$

Let us now derive the formulas for the calculation and analysis of the dynamic properties of combined control systems. As in Chapter Four, we intend to make use of the D-decomposition curve for the gain of the degenerate part of the equation of each control subsystem.

The gain in the degenerate part of each subsystem is made up of two factors, the plant gain K_i and the gain μ_i of the unstabilized section. The gain $K_{\deg i}$ of the degenerate equation can obviously be altered by changing μ_i. We will write $K_{\deg i}=\mu_i K_i$, and hence

$$b_{ii}(p)=D_i(p)F_{ni}(p)Q_i(p)\theta_{mi}(p)+\mu_i K_i F_{mi}(p)\theta_{mi}(p)=$$
$$=b_i(p)+K_{\deg i}F_{mi}(p)\theta_{mi}(p).$$

In this nomenclature equations (5.43)—(5.45) take the form

$$A_{11b}=\begin{vmatrix} b_2(p)+\mu_2K_2F_{m2}(p)\theta_{m2}(p) & d_{33}(p)\alpha_{32} \\ d_{22}(p)\alpha_{23} & b_3(p)+\mu_3K_3F_{m3}(p)\theta_{3m}(p) \end{vmatrix}, \tag{5.46}$$

$$A_{12b}=\begin{vmatrix} d_{11}\alpha_{12} & d_{33}(p)\alpha_{32} \\ d_{11}(p)\alpha_{13} & \mu_3K_3+K_{\deg 3}F_{m3}(p)\theta_{3m}(p) \end{vmatrix}, \tag{5.47}$$

$$A_{13b}=\begin{vmatrix} d_{11}(p)\alpha_{12} & \mu_2K_2+K_{2\deg}F_{m2}(p)\theta_{2m}(p) \\ d_{11}(p)\alpha_{13} & d_{22}(p)\alpha_{23} \end{vmatrix}. \tag{5.48}$$

Interchanging the rows and the columns in (5.42), we expand the transposed determinant in elements of the first row. Using our nomenclature we thus write

$$\Delta_{3b}=b_1(p)A_{11b}(p)+\mu_1K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)-$$
$$-d_{22}(p)\alpha_{21}A_{12b}(p)+d_{33}\alpha_{31}A_{13b}(p), \tag{5.49}$$

and the equation of the D-decomposition curve for μ_1 is

$$\bar{\mu}_1=-\frac{b_1(p)A_{11b}(p)-d_{22}(p)\alpha_{21}A_{12b}(p)+d_{33}(p)\alpha_{31}A_{13b}(p)}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)}, \tag{5.50}$$
$$p=j\omega.$$

We divide the numerator and the denominator of (5.41) by $K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)$. After elementary manipulations we obtain

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{L(p) - B(p) + D(p)}{\sigma(p)}, \tag{5.51}$$

where

$$L(p) = \frac{A_{11b}(p)\left\{l_{11}(p) + [\rho_{11}(p) + d_{11}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[f_1(p) + \beta_{12}f_2(p) + \beta_{13}f_3(p)]\right\}}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$B(p) = \frac{A_{12b}(p)\left\{l_{12}(p)\frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)} + [\rho_{22}(p) + d_{22}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[\beta_{21}f_1(p) + f_2(p) + \beta_{23}f_3(p)]\right\}}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$D(p) = \frac{A_{13b}(p)\left\{\frac{Y_{3\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}l_{33}(p) + [\rho_{33}(p) + d_{33}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[\beta_{31}f_1(p) + \beta_{32}f_2(p) + f_3(p)]\right\}}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$\sigma(p) = \mu_1 + \frac{b_1(p)A_{11b}(p) - d_{22}(p)\alpha_{21}A_{12b}(p) + d_{33}(p)\alpha_{31}A_{13b}(p)}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$p = j\omega.$$

This equation can be readily generalized for an n-variable system. It may be used as a working formula in stability calculations and in selecting system parameters that ensure the required performance characteristics.

Indeed, the denominator in (5.51) is the sum of the gain μ_1 and the corresponding D-decomposition curve. If the D-decomposition curve is available (from which the stability of the entire system can be inferred), the well-known rule /39/ can be applied to directly determine from this curve the values of the denominator in (5.51) at any frequency. The numerator of (5.51) is the equation of the auxiliary curve. The dynamic properties of the entire multivariable system are completely determined by the position of the D-decomposition curve and the auxiliary curve.

As an example we consider a two-variable combined control system, from which we will try to deduce some general properties of combined control systems.

For $n = 2$ equation (5.51) is written as

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{L(p) + N(p) + P(p)}{\sigma(p)}, \qquad p = j\omega, \tag{5.52}$$

where

$$L(p) = \frac{A_{11b}(p)\,l_{11}(p)}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$N(p) = \frac{A_{11b}(p)[\rho_{11}(p) + d_{11}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[f_1(p) + \beta_{12}f_2(p)]}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$P(p) = \frac{A_{12b}(p)\,l_{22}(p)\frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)} + [\rho_{22}(p) + d_{22}(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)}[\beta_{21}f_1(p) + f_2(p)]}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)},$$

$$\sigma = \mu_1 + \frac{b_1(p)A_{11b}(p) - d_{22}(p)\alpha_{21}}{K_1F_{m1}(p)\theta_{m1}(p)A_{11b}(p)}.$$

Inserting for the operators in (5.52) their expressions from (5.5), we obtain

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{\mu_1[1+\zeta(p)]+\gamma(p)+\delta(p)-\xi(p)}{\varphi(p)}. \tag{5.53}$$

where

$$\zeta(p) = \frac{\theta_{n1}(p)}{\theta_{m1}(p)}[f_1(p)+\beta_{12}f_2(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)},$$

$$\gamma(p) = \frac{F_{n1}(p)\,Q_1(p)}{F_{m1}(p)}[f_1(p)+\beta_{12}f_2(p)]\frac{1}{Y_{1\,\mathrm{ref}}(p)},$$

$$\delta(p) = K_1F_{n1}(p)\,Q_1(p)\,\theta_{m1}(p)\,\alpha_{12}\left\{K_2\mu_2F_{2m}(p)\,\theta_{m2}(p)\frac{Y_{2\,\mathrm{ref}}(p)}{Y_{1\,\mathrm{ref}}(p)}+\right.$$

$$\left.+[K_2\mu_2F_{m2}(p)\,\theta_{n2}(p)+K_2F_{n2}(p)\,Q_2(p)\,\theta_{m2}(p)]\times\frac{1}{Y_{1\,\mathrm{ref}}}[\beta_{21}f_1(p)+f_2(p)]\right\},$$

$$\xi(p) = K_1F_{m1}(p)\,\theta_{m1}(p)\,[K_2\mu_2F_{m2}(p)\,\theta_{m2}(p)+D_2(p)\,Q_2(p)\,F_{n2}(p)\,\theta_{m2}(p)$$

and

$$\varphi(p) = \frac{D_1(p)\,Q_1(p)\,F_{n1}(p)\,\theta_{m1}(p)\,[P_1]-K_2F_{n2}(p)\,Q_2(p)\,\theta_{m2}(p)\,\alpha_{21}}{P_2}+\mu_1,$$

where

$$P_1 = [\mu_2K_2F_{m2}(p)\,\theta_{m2}(p)+D_2(p)\,Q_2(p)\,F_{n2}(p)\,\theta_{m2}(p)],$$

$$P_2 = K_1F_{m1}(p)\,\theta_{m1}(p)\,[K_2\mu_2F_{m2}(p)\,\theta_{m2}(p)+D_2(p)\,Q_2(p)\,F_{n2}(p)\,\theta_{m2}(p)].$$

The following conclusions follow from (5.53).

1. Suppose that the controlled variables are coupled neither through the plant nor through the load, i.e., $\alpha_{12}=0$ and $\beta_{12}=\beta_{21}=0$. Then

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{\mu_1+\mu_1\dfrac{\theta_{n1}(p)}{\theta_{m1}(p)}\dfrac{f_1(p)}{Y_{1\,\mathrm{ref}}(p)}}{\mu_1+\dfrac{D_1(p)\,Q_1(p)\,F_{n1}(p)}{F_{m1}(p)\,K_1}} \qquad (p=j\omega) \tag{5.54}$$

and in the absence of load disturbance we find

$$\frac{Y_1(p)}{Y_{1\,\mathrm{ref}}(p)} = \frac{\mu_1}{\mu_1+\dfrac{D_1(p)\,Q_1(p)\,F_{n1}(p)}{K_1F_{m1}(p)}}. \tag{5.55}$$

We see that the auxiliary curve is sensitive to load signals. The function $\frac{\theta_{n1}(p)}{\theta_{m1}(p)}$ can be so chosen as to ensure a desirable transient. As it could have been expected, the function $\frac{\theta_{n1}(p)}{\theta_{m1}(p)}$ does not influence the stability of the system. If $\alpha_{12}=0$, $\beta_{12}=\beta_{21}\neq 0$, the system dynamics can be improved by supplementing the self-load with load from other subsystems.

2. The system dynamics are invariably determined by the D-decomposition curve and the auxiliary curve. The auxiliary curve is highly sensitive to the function $\frac{\theta_{n1}(p)}{\theta_{m1}(p)}$. Hence follows a very significant conclusion, namely that the system dynamics can be altered between wide limits with the aid of transducers $\frac{\theta_{n1}(p)}{\theta_{m1}(p)}$.

3. The equation of the D-decomposition curve can be written in an alternative form. Carrying out term-by-term division in the second term in the denominator of (5.53), we obtain for the D-decomposition curve

$$\bar{\mu}_1 = -\frac{D_1(p)\,Q_1(p)\,F_{n1}(p)}{K_1 F_{m1}(p)} + \frac{K_2}{K_1}\frac{F_{n2}(p)}{F_{n1}(p)} \times \\ \times \frac{\theta_{m2}(p)\,Q_2(p)\,\alpha_{21}(p)}{\theta_{m1}(p)\,[K_2\mu_2 F_{m2}(p)\,\theta_{m2}(p) + D_2(p)\,Q_2(p)\,F_{n2}(p)\,\theta_{m2}(p)]}. \tag{5.56}$$

The first term in equation (5.56) is the D-decomposition curve for μ_1 in the uncoupled case, the second term gives the contribution from coupling.

Equation (5.56) can be written in a still different form:

$$\bar{\mu}_1 = D_{\mu_1} + K_1 \frac{F_{n2}(p)}{F_{n1}(p)} \cdot \frac{Q_2(p)\,\alpha_{21}(p)}{\theta_{m1}(p)\,F_{m2}(p)\,[\mu_2 - D_{\mu_2}]}, \tag{5.57}$$

where

$$D_{\mu_2} = -\frac{D_2(p)\,Q_2(p)\,F_{n2}(p)}{K_2 F_{m2}(p)}$$

is the D-decomposition curve for μ_2 (the gain of the second subsystem). D-decomposition curves for μ_1 and μ_2 can be plotted before we have actually decided what $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ are to be used.

D-decomposition curves enable us to determine all the terms in equation (5.57), with the exception of the transfer function $\theta_{mi}(p)$ which is chosen in compliance with a certain quality criterion of the entire system. We have thus derived a formula for the synthesis of combined control systems.

The method described in this section can obviously be applied to systems with n controlled variables as well.

§ 5.6. LOAD REJECTION

The effects of load and other disturbances are dealt with in separate sections of the following chapters. At this stage of our discussion of combined control system dynamics, however, we cannot ignore this problem altogether. It should be emphasized that the load rejection* is a characteristic feature of combined control systems.

Since load rejection is generally related to the problem of coarseness in the sense of A. A. Andronov, we shall investigate the complete, and not the degenerate equation. Suitable working formulas for load rejection in the degenerate equation will be given at a later stage.

Consider the equation of the j-th controlled variable in a system with n variables which are coupled through the plant (equation (5.15)). We see from (5.15) that the load does not affect the controlled variable if

$$A_{jl}(p)\,[K_{cl}(\rho_{ll}(p) + d_{ll}(p)) + N_{ll}(p)] = 0. \tag{5.58}$$

* We are dealing with loads applied to the plant only.

Equality (5.58) is satisfied if

$$\left.\begin{array}{l} \text{(a)}\ A_{jl}(p)=0 \\ \text{or} \\ \text{(b)}\ K_{ci}(\rho_{il}(p)+d_{il}(p))+N_{il}(p)=0. \end{array}\right\} \qquad (5.59)$$

It is readily seen, however, that the requirement $A_{ji}(p) = 0$ is inadmissible, since, as it follows from (5.15), $Y_{k\,\mathrm{ref}}$ is also multiplied by $A_{ij}(p)$, and the condition $A_{ij}(p) = 0$ would eliminate the entire control system, as well as the load. *

Thus load rejection is based on the condition

$$K_{ci}[\rho_{il}(p)+d_{il}(p)]+N_{il}(p)=0. \qquad (5.59')$$

Inserting for the operators in (5.59') their expressions from (5.5), we find

$$K_{ci}[K_i\mu_i F_{mi}(p)\theta_{ni}(p)+K_i F_{ni}(p)Q_i(p)\theta_{mi}(p)]+ \\ +K_i Q_i(p)\theta_{mi}(p)R_i(p)F_{mi}(p)=0, \qquad (5.60)$$

or

$$K_{ci}\left[\frac{\theta_{ni}(p)}{\theta_{mi}(p)}\frac{\mu_i}{Q_i(p)}+\frac{F_{ni}(p)}{F_{mi}(p)}\right]+R_i(p)=0,$$

whence follows an expression for the transducer ratio:

$$\frac{\theta_{ni}(p)}{\theta_{mi}(p)}=-\left(R_i(p)+K_{ci}\frac{F_{ni}(p)}{F_{mi}(p)}\right)\frac{Q_i(p)}{\mu_i}= \\ =-\frac{(R_i(p)F_{mi}(p)+K_{ci}F_{ni}(p))Q_i(p)}{\mu_i F_{mi}(p)}. \qquad (5.61)$$

This function is fairly difficult to implement since, as it follows from (5.61), the degree of $\theta_{ni}(p)$ should be greater than the degree of $\theta_{mi}(p)$ at least by an amount equal to the degree of the product $R_i(p)Q_i(p)$. This is precisely the degree of the section which includes the stabilized component $R_i(p)$ and the unstabilized component $Q_i(p)$.

The problem, however, is solved very easily by a simple modification of structure. Clearly, the effect of load is eliminated if the output of $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ is delivered directly to the plant input. Indeed, first we write equation (5.61) in a different form:

$$\frac{K_{ci}}{R_i(p)}\frac{\mu_i}{Q_i(p)}\frac{\theta_{ni}(p)}{\theta_{mi}(p)}+\frac{K_{ci}}{R_i(p)}\frac{F_{ni}(p)}{F_{mi}(p)}+1=0. \qquad (5.62)$$

Now the transducer output is delivered directly to the plant input. In the result of this operation, the factor before $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ should be divided by the transfer functions of those elements which are dispensed with in the new

* This problem is considered in more detail in the next chapter.

configuration, i.e., the transfer functions $\frac{K_{ci}}{R_i(p)}\frac{\mu_i}{Q_i(p)}$. We thus obtain for the transducer ratio

$$\frac{\theta_{ni}(p)}{Q_i(p)} = -\left[\frac{K_{ci}}{R_i(p)}\frac{F_{ni}(p)}{F_{mi}(p)} + 1\right]. \tag{5.63}$$

This transfer function can be implemented without difficulty.

As we have already noted, $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ does not influence the system stability, provided that $\theta_{mi}(p)$ has left-half-plane roots only. The problem of Andronov's coarseness therefore does not arise in this case.

§ 5.7. LOAD REJECTION FOR $K_c \to \infty$

In the preceding section we showed how to choose the transducer ratio and how to connect the transducer to the system so as to ensure complete load rejection. Earlier we demonstrated the advisability of using structures which are stable for $K_{ci} \to \infty$. In this section we correspondingly proceed to consider the choice of $\frac{\theta_{ni}(p)}{\theta_{mi}(p)}$ for this special class of structures, when the fundamental dynamic properties of the system are entirely specified by the degenerate equation. Our aim, of course, is to achieve perfect load rejection.

From (5.46) it is clear that the load does not affect system dynamics if

$$\rho_{ii}(p) + d_{ii}(p) = 0. \tag{5.64}$$

Inserting for $\rho_{ii}(p)$ and $d_{ii}(p)$ their values from (5.5), we have

$$K_i\mu_i F_{mi}(p)\,\theta_{ni}(p) + K_i F_{ni}(p)\,Q_i(p)\,\theta_{mi}(p) = 0$$

or

$$\frac{\mu_i}{Q_i(p)}\frac{\theta_{ni}(p)}{\theta_{mi}(p)} = -\frac{F_{ni}(p)}{F_{mi}(p)}. \tag{5.65}$$

This can be achieved without difficulty if the structure is appropriately modified. Indeed, if the transducer output is connected as shown in Figure 5.2 (after the element with the transfer function $\frac{\mu_i}{Q_i(p)}$), the left-hand side of (5.65) is divided by $\frac{\mu_i}{Q_i(p)}$, and for $K_{ci} \to \infty$ we finally have

$$\frac{\theta_{ni}(p)}{\theta_{mi}(p)} = \frac{F_{ni}(p)}{F_{mi}(p)}. \tag{5.66}$$

There is no need to emphasize that a transducer with this transfer function can be built without any difficulty.

In the general case of an n-variable system, load rejection is achieved by using structure configurations shown in Figure 5.2. The transducers are chosen from the condition

$$\frac{\theta_{ni}}{\theta_{mi}} = -\frac{F_{ni}}{F_{mi}}, \tag{5.67}$$

where $i = 1, 2, \ldots, n$.

In practice, the system can be further simplified by using a stabilizer which doubles as a transducer. In this case the load disturbances $\sum_{k=1}^{n} \beta_{ik} f_k$ are delivered directly to the stabilizer input, as is shown in Figure 5.3. There is no need to adjust the parameters of the stabilizer and the transducer to achieve matching, as there is only one set of parameters in question, the parameters of the stabilizer.

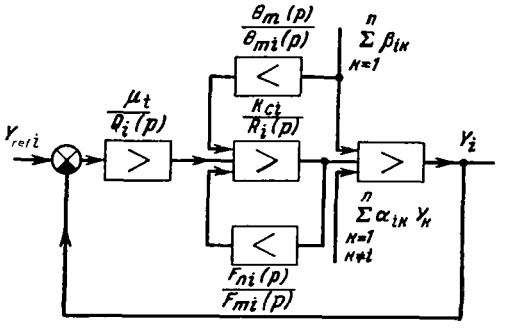

FIGURE 5.2. A combined control system with load rejection.

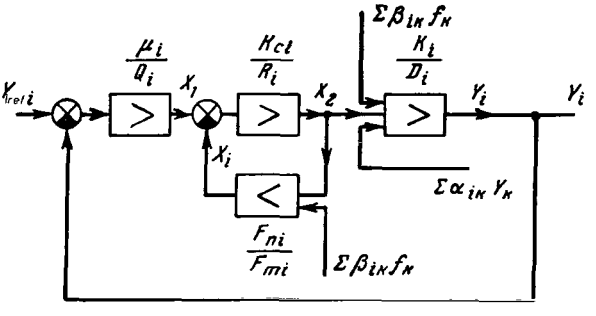

FIGURE 5.3. A stabilizer used for load rejection.

Thus, for sufficiently large gain K_{ci}, the structure in Figure 5.3 ensures that the control process is independent of loads and disturbances applied to the plant. (The case of disturbances, loads, and other interferences applied not to the plant but elsewhere in the system will be considered separately.)

Chapter Six

INVARIANCE AND NONINTERACTION IN MULTIVARIABLE CONTROL SYSTEMS

§ 6.1. INTRODUCTORY REMARKS

In automatic control systems one always faces the problem of eliminating the effect of disturbances (loads) on the variation (or, in particular cases, constancy) of the controlled variables. In other words, we have to deal with rejection of external disturbances acting on the control system.

The principle of invariance, when some generalized coordinate of a dynamic system is independent of disturbances, was formulated by N. N. Luzin in 1940, as a generalization of the previous results of G. V. Shchipanov. The problem of invariance was subsequently developed by Kulebakin /26, 27/, Petrov /51/, Fel'dbaum /66/, Kukhtenko /24/, and others (see Bibliography). Rozonoer /58/ reduced the problem of invariance to a variational problem.

Synthesis of systems where the controlled variable is entirely independent of external disturbances, the so-called perfect load-rejecting systems, is discussed by Shchipanov /76/. In later researches /27, 31, 32, 51, 56/, Shchipanov's results were considered in very great detail and we now have a thorough understanding of his fundamental contributions, as well as of some inaccuracies in his work.

The practical significance of Shchipanov's ideas is due to the fact that the load-rejection principle is realizable in real systems. Petrov /51/ formulated the two-channel principle, which provides us with a key to the design of single-variable systems with complete or partial rejection of external disturbances.

In this chapter invariance is considered in application to multivariable control systems. The characteristic problems of multivariable control systems, aside from those which, though solved by the general methods, refer to single-variable systems, arise from the fact that each controlled variable is influenced not only by various disturbances but also by all the other controlled variables: all the variables interact through the plant, the measurement devices (in multidimensional servosystems), and the load.

One of the fundamental problems in multivariable control is the problem of noninteraction, i. e., choice of structures and system parameters ensuring that the various controlled variables do not interact, so that the control subsystems for each variable can be considered independently of all the rest.

The problem of noninteraction was first formulated by I.N. Voznesenskii /10, 11/ and he was the first to propose methods for the selection of regulator connections that ensured active control by each subsystem. Noninteraction is the subject of numerous Soviet /5, 6, 21, 19, 52/ and Western publications (see Bibliography at the end of the book).

The problem of noninteraction is closely related to the problem of invariance. It is shown /27, 29, 51/ that the noninteraction conditions sometimes coincide with invariance criteria.

It is clear from the results of Chapter Five that rejection of external disturbances does not ensure noninteraction. Our task is thus to consider the relationship between invariance and noninteraction. It will be shown that invariance in relation to external disturbances does not automatically ensure noninteraction and vice versa: noninteraction does not automatically mean invariance. That these problems should be considered separately follows from certain physical realizability conditions, and in particular from conditions of stability of the entire multivariable system.

§ 6.2. THE PROBLEM OF NONINTERACTION

In noninteracting multivariable systems, the controlled variation of one of the variables does not influence the other variables. Noninteraction in this sense may be complete (or perfect) or alternatively it may hold true to a certain finite accuracy.

Noninteraction can be considered from two points of view. First, it may be attributed to what we call technological factors. As an example, take the system of frequency and speed control in an asynchronous motor. Desirable performance characteristics, especially when starting or stopping the motor, are ensured by varying the stator voltage and the supply current frequency according to equations which differ from the natural variation of the variable-frequency outputs (e.g., in a variable-speed synchronous generator).

Second, noninteraction may be regarded as a certain dynamic property of the system, an organic outgrowth of its structure. This case is of considerable importance and will be treated separately in the following. I will first discuss the fundamental results obtained by Voznesenskii /10, 11/ and American authors /77/ and then proceed to analyze my own contributions to the subject.

1. Voznesenskii's fundamental results /10, 11/

These results deserve special attention, as they were essentially the first contributions to the theory of automatic control and laid down a foundation for the design of quality control systems /10, 11, 19, 52/. They also provided a point of departure for numerous later researches.

Voznesenskii's results apply to cases when the controlled variables interact through the plant only. The problem is thus stated as follows: choose a control system such that noninteraction of the individual controlled variables is ensured.

The problem is investigated for a multivariable control system described by the following set of differential equations:

$$\left.\begin{aligned} a_1 \frac{dy_1}{dt} &= \mu_1 - \mu_{10}, \\ &\cdots\cdots\cdots \\ a_n \frac{dy_n}{dt} &= \mu_n - \mu_{n0}, \end{aligned}\right\} \tag{6.1}$$

where a_i are constants, y_i are the controlled variables, μ_{i0} are the steady-state loads, torques, etc., μ_i are the loads, torques, etc., corresponding to the variables y_i.

The quantities μ_i are controlled by the controllers m_i. To ensure noninteraction, μ_i is controlled not by the i-th controller alone but by all the interconnected regulators jointly. The behavior of the controllers is thus described by the following set of equations:

$$\left.\begin{aligned} \mu_1 &= k_{11}m_1 + k_{12}m_2 + \ldots + k_{1n}m_n, \\ \mu_2 &= k_{21}m_1 + k_{22}m_2 + \ldots + k_{2n}m_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ \mu_n &= k_{n1}m_1 + k_{n2}m_2 + \ldots + k_{nn}m_n, \end{aligned}\right\} \tag{6.2}$$

where k_{ik} gives the effect of the k-th controller on the i-th parameter μ_i.

In a system with μ controlled variables a measuring device (a sensor) is provided for each variable. Ideal transducers are assumed, satisfying the relations

$$y_i = n_i z_i, \tag{6.3}$$

where $i = 1, 2, \ldots, n$. The various controllers are described by the equations

$$\left.\begin{aligned} m_1 &= l_{10} + l_{11}z_1 + l_{12}z_2 + \ldots + l_{1n}z_n, \\ m_2 &= l_{20} + l_{21}z_1 + l_{22}z_2 + \ldots + l_{2n}z_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ m_n &= l_{n0} + l_{n1}z_1 + l_{n2}z_2 + \ldots + l_{nn}z_n. \end{aligned}\right\} \tag{6.4}$$

Here l_{ik} are the transfer numbers between the measurement devices z_k and the controllers m_i. These are the numbers to be determined if noninteraction is to be ensured.

Inserting for m_i in (6.2) their expressions from (6.4), we find

$$\left.\begin{aligned} \mu_1 &= k_{11}l_{11} + k_{11}l_{11}z_1 + k_{11}l_{12}z_2 + k_{11}l_{13}z_3 + \ldots + k_{11}l_{1n}z_n + \\ &\quad + k_{12}l_{20} + k_{12}l_{21}z_1 + k_{12}l_{22}z_2 + k_{12}l_{23}z_3 + \ldots + k_{12}l_{2n}z_n + \\ &\quad + \ldots + k_{13}l_{30} + k_{13}l_{31}z_1 + k_{13}l_{32}z_2 + k_{13}l_{33}z_3 + \ldots + \\ &\quad + k_{13}l_{3n}z_n + \ldots = \sum_{i=1}^{n} k_{1i}l_{i0} + z_1\sum_{1}^{n} k_{1i}l_{i1} + z_2\sum_{1}^{n} k_{1i}l_{i2} + \ldots \\ &\qquad\qquad \ldots + z_n\sum_{1}^{n} k_{1i}l_{in}, \\ \mu_2 &= \sum_{i=1}^{n} k_{2i}l_{i0} + z_1\sum_{1}^{n} k_{2i}l_{i1} + z_2\sum_{1}^{n} k_{2i}l_{i2} + \ldots + z_n\sum_{1}^{n} k_{2i}l_{in}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_n &= \sum_{i=1}^{n} k_{ni}l_{i0} + z_1\sum_{1}^{n} k_{ni}l_{i1} + z_2\sum_{1}^{n} k_{ni}l_{i2} + \ldots + z_n\sum_{1}^{n} k_{ni}l_{in}. \end{aligned}\right\} \tag{6.5}$$

The coupling coefficients l_{ik} are chosen so that all the sums of the products $k_{ps}l_{sq}$, $p \neq q$, vanish, i.e.,

$$\sum_{s=1}^{n} k_{ps}l_{sq} = 0 \quad \text{for} \quad q \neq p. \tag{6.6}$$

In (6.5) there are $n(n+1)$ unknown coefficients, whereas (6.6) provides only $n(n-1)$ equations for these coefficients (n coefficients of the form l_{i0} and n coefficients of the form k_{ii} are not included). The number of missing equations is thus

$$(n+1)n - n(n-1) = n^2 + n - n^2 + n = 2n.$$

The $2n$ missing equations can be obtained from the following conditions. Making use of (6.6), we write (6.5) in the form

$$\left.\begin{aligned} \mu_1 &= \sum_{l=1}^{n} k_{1s}l_{s0} + z_1 \sum_{1}^{n} k_{1s}l_{s1}, \\ \mu_2 &= \sum_{1}^{n} k_{2s}l_{s0} + z_2 \sum_{1}^{n} k_{2s}l_{s2}, \\ &\cdots\cdots\cdots\cdots \\ \mu_n &= \sum_{1}^{n} k_{ns}l_{s0} + z_n \sum_{1}^{n} k_{ns}l_{sn}. \end{aligned}\right\} \tag{6.7}$$

If the control domain for the given range of μ_{i0} is denoted by

$$\Delta\mu_l = \mu_{l\,\max} - \mu_{\min}, \tag{6.8}$$

the coordinates of the measurement devices by $z_{l\,\min}$ and $z_{i\,\max}$ respectively, and the irregularity coefficients by

$$\delta_i = \frac{y_{l\,\max} - y_{l\,\min}}{y_{l0}}, \tag{6.9}$$

we can make use of (6.3) to obtain after simple manipulations

$$\left.\begin{aligned} \Delta\mu_1 &= -\frac{\delta_1 y_{10}}{n_1} \sum_{1}^{n} k_{1s}l_{s1}, \\ \Delta\mu_2 &= -\frac{\delta_2 y_{20}}{n_2} \sum_{1}^{n} k_{2s}l_{s2}, \\ &\cdots\cdots\cdots\cdots \\ \Delta\mu_n &= -\frac{\delta_n y_{n0}}{n_n} \sum_{1}^{n} k_{ns}l_{sn}. \end{aligned}\right\} \tag{6.10}$$

If $\Delta\mu_i$ and δ_i are known, the set (6.10) provides n additional equations. Now the last n missing equations are obtained by substituting the steady-state values μ_{i0} that correspond to the steady-state controlled variables y_{i0} for μ_{i0} in (6.7) and remembering that $z_{i0} = \frac{y_{i0}}{n_i}$:

$$\left.\begin{aligned} \mu_{10} &= \sum_{s=1}^{n} k_{1s}l_{s0} + \frac{y_{10}}{n_1} \sum_{s=1}^{n} k_{1s}l_{s1}, \\ \mu_{20} &= \sum_{1}^{n} k_{2s}l_{s0} + \frac{y_{20}}{n_2} \sum_{1}^{n} k_{2s}l_{s2}, \\ &\cdots\cdots\cdots\cdots \\ \mu_{n0} &= \sum_{1}^{n} k_{ns}l_{s0} + \frac{y_{n0}}{n_n} \sum_{1}^{n} k_{ns}l_{sn}. \end{aligned}\right\} \tag{6.11}$$

Equations (6.6), (6.10), and (6.11) give $n(n+1)$ equations in $n(n+1)$ unknowns, $l_{10}, \ldots, l_{1n}, l_{20}, \ldots, l_{2n}, \ldots, l_{nn}$.

Solving these equations for l_{ij}, we find

$$l_{ij} = -\frac{\Delta\mu_j n_j}{\delta_j y_{j0}} \frac{\Delta_{ji}}{\Delta} \qquad (j = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, n), \tag{6.12}$$

where

$$\Delta = \begin{vmatrix} k_{11} & k_{12} & \ldots & k_{1n} \\ k_{21} & k_{22} & \ldots & k_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ k_{j1} & k_{j2} & \ldots & k_{jn} \\ k_{n1} & k_{n2} & \ldots & k_{nn} \end{vmatrix}$$

and Δ_{ji} is the cofactor of k_{ji}. Having thus selected the transfer numbers l_{ik}, we ensure noninteraction within the framework of our assumptions. Noninteraction is thus ensured if the following are known:

(a) the matrix of the coefficients $\|k_{ij}\|$, where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$;

(b) the irregularity coefficients δ_i, where $i = 1, 2, \ldots, n$;

(c) the normal loads μ_{i0}, where $i = 1, 2, \ldots, n$;

(d) the rated (guaranteed) steady-state values of the controlled variables y_{i0}, where $i = 1, 2, \ldots, n$.

It follows from the preceding that noninteraction is primarily the outcome of a certain mode of operation. In a different operating mode, interaction may be restored.

We have discussed here the fundamental contributions of Voznesenskii. Later researches are based on his work, and their aim is to establish noninteraction criteria under more complex conditions. Note that Voznesenskii's presentation is most elementary, since each controlled variable is described by a first-order differential equation. Furthermore, ideal transducers (measuring devices) and controllers are assumed.

The noninteraction conditions become obviously more complicated in slightly more complex systems. It has been shown /19, 52/ that in indirect control systems employing the same simple plant as before, noninteraction requires equality of time constants of all the servomotors.

We do not discuss here further developments of Voznesenskii's approach, and the reader is referred to special literature /5, 6, 16, 19, 52, etc./.

2. The method of Boksenbom and Hood /77/

Boksenbom and Hood /77/ published their results in 1949 and they are essentially similar to those of Voznesenskii. The only difference is the mathematics of the solution. Almost all later Western publications in this field /81, 82, 77, 78, 79, etc./ are based on the original paper of Boksenbom and Hood*, and we therefore proceed with a detailed analysis of their method. Consider a plant with i controlled (dependent) variables y, which are henceforth referred to as output variables or briefly outputs, and n independent inputs x (Figure 6.1).

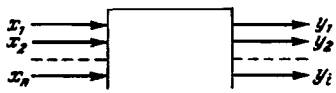

FIGURE 6.1. A multivariable controlled object with n inputs and i outputs.

* With the exception of the book by Mesarović /85/, mentioned in Chapter Three.

If each output is dependent on all the inputs, we may write the following set of equations in Laplace transforms:

$$\left.\begin{aligned} Y_1 &= E_{11}X_1 + E_{12}X_2 + \ \dots\ + E_{1n}X_n, \\ Y_2 &= E_{21}X_1 + E_{22}X_2 + \ \dots\ + E_{2n}X_n, \\ &\dots\dots\dots\dots\dots\dots \\ Y_i &= E_{i1}X_1 + E_{i2}X_2 + \ \dots\ + E_{in}X_n \end{aligned}\right\} \tag{6.13}$$

or

$$Y_i = \sum_{k=1}^{n} E_{ik}X_k, \qquad i = 1, 2, \dots, i.$$

The operators E_{ik} are the transfer functions between the k-th input and the i-th output.

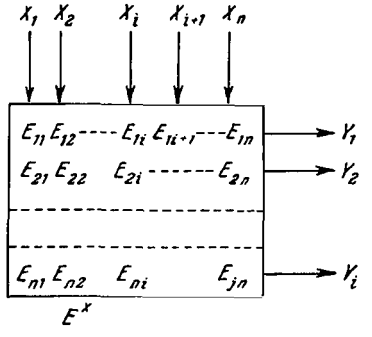

FIGURE 6.2. Schematic representation of the controlled object (see Figure 6.1).

Equation (6.13) may be written in matrix form (Figure 6.2). Each matrix element E_{ik} stands for the corresponding transfer function. Each input as if acts on its own column and, upon multiplication by the elements of that column, gives the output sum in the corresponding row. Thus X_1 acts on the first column: it is multiplied by each element of that column, the products are added up, and the sum is the output written in the first row. From the general matrix E we isolate the first i columns, this being the number of dependent (controlled) outputs, and form an $i \times i$ square matrix.

It is assumed that with n inputs only $i<n$ outputs are controlled. For this reason $n-i$ inputs can be manipulated as desired.

Figure 6.3 is a functional diagram of a control system for a single variable. The controller outputs X_i are represented by the set of equations

$$\left.\begin{aligned} \tilde{X}_1 &= c_{11}(Y_1 - \bar{Y}_1) + c_{12}(Y_2 - \bar{Y}_2) + \ \dots\ + c_{1i}(Y_i - \bar{Y}_i) + \\ &\quad + c'_{1\,(i+1)}(X_{i+1} - \bar{X}_{i+1}) + \ \dots\ + c'_{1n}(X_n - \bar{X}_n), \\ \tilde{X}_2 &= c_{21}(Y_1 - \bar{Y}_1) + c_{22}(Y_2 - \bar{Y}_2) + \ \dots\ + c_{2i}(Y_i - \bar{Y}_i) + \\ &\quad + c'_{2\,(i+1)}(X_{i+1} - \bar{X}_{i+1}) + \ \dots\ + c'_{2n}(X_n - \bar{X}_n), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \tilde{X}_n &= c_{n1}(Y_1 - \bar{Y}_1) + c_{n2}(Y_2 - \bar{Y}_2) + \ \dots\ + c_{ni}(Y_i - \bar{Y}_i) + \\ &\quad + c'_{n\,(i+1)}(X_{i+1} - \bar{X}_{i+1}) + \ \dots\ + c'_{nn}(X_n - \bar{X}_n). \end{aligned}\right\} \tag{6.14}$$

First note that the controller outputs depend not only on the deviations of the respective controlled variables but also on the deviations of all the other variables, and equations (6.14) are analogous in this sense to equations (6.2) in Voznesenskii's method. The only difference is that contributions from the slack inputs X_{l+1}, etc., are added.

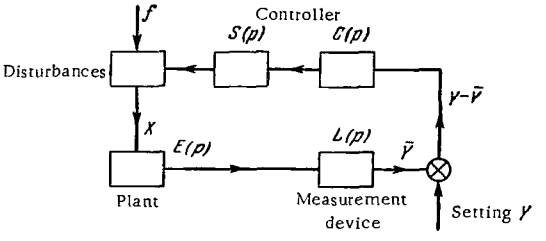

FIGURE 6.3. A single-variable control system.

An obviously interesting approach is to use the slack inputs as additional control factors. Equations (6.14) can be written in the following abbreviated form:

$$\tilde{X}_k = \sum_{j=1}^{l} C_{kj}(Y_l - \bar{Y}_l) + \sum_{\mu=l+1}^{n} C'_{k\mu}(X_\mu - \bar{X}_\mu). \qquad (6.15)$$

Figure 6.4 is a matrix representation of (6.15). The controller matrix is interpreted in the same way as the plant matrix in Figure 6.2. The inputs $Y-\bar{Y}$ and $X-\bar{X}$ act on the columns and the row outputs are $\tilde{X}$. Each input is multiplied by all the column elements, and the sum of these products in each row gives the corresponding output.

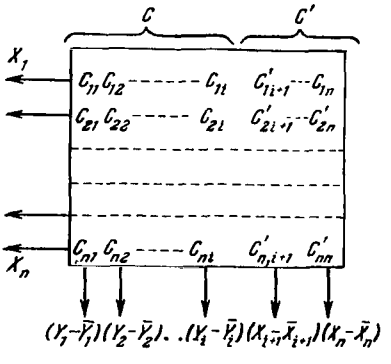

FIGURE 6.4. Schematic representation of a controller.

A complete control system is obtained when the previous equations are supplemented with the equations of measurement devices and servomechanisms.

For the measurement devices we may write

$$\left.\begin{array}{c} \bar{Y}_\nu = L_{\nu\nu} Y_\nu \\ (\nu = 1, 2, \ldots, i), \\ \bar{X}_\mu = L_{\mu\mu} X_\mu \\ (\mu = i+1, i+2, \ldots, n). \end{array}\right\} \tag{6.16}$$

In what follows it is generally assumed that each independent variable has its own servo, which is actuated by the corresponding signal $\tilde{X}$. Introducing the disturbances f_k, we write

$$X_k = s_{kk} \tilde{X}_k + f_k \qquad (k = 1, 2, \ldots, n). \tag{6.17}$$

We have thus obtained the following set of equations for a multivariable control system

The plant equation

$$Y_j = \sum_{k=1}^{n} E_{jk} X_k \qquad (j = 1, 2, \ldots, n).$$

The controller equation

$$\tilde{X}_k = \sum_{\nu=1}^{i} c_{k\nu} (Y_\nu - \bar{Y}_\nu) + \sum_{\mu=i+1}^{n} c'_{k\mu} (X_\mu - \bar{X}_\mu) (k = 1, 2, \ldots, n). \tag{6.18}$$

The equation of the measurement device

$$\bar{Y}_\nu = L_{\nu\nu} Y_\nu, \qquad \bar{X}_\mu = L_{\mu\mu} X_\mu$$
$$(\nu = 1, 2, \ldots, i; \ \mu = i+1, \ldots, n).$$

Servo and disturbances

$$X_k = s_{kk} \tilde{X}_k + f_k \qquad (k = 1, 2, \ldots, n).$$

This set can be represented, as before, in matrix form. If the plant has three controlled variables (dependent outputs) and five independent inputs, the corresponding matrix is shown in Figure 6.5.

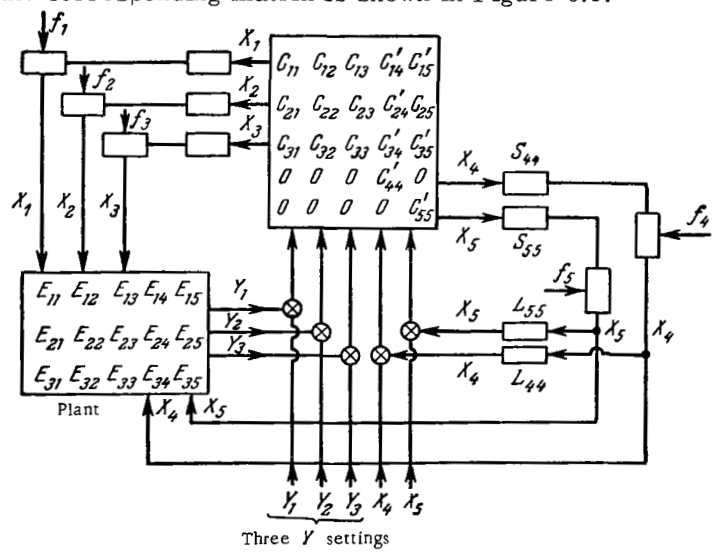

FIGURE 6.5. Matrix representation of plant and controller.

Solving (6.18) for the controlled variable Y_i, we find

$$Y_j = \sum_{k=1}^{n} \sum_{\nu=1}^{i} E_{jk} S_{kk} C_{k\nu} (Y_\nu - L_{\nu\nu} Y_\nu) + \\ + \sum_{k=1}^{n} \sum_{\mu=i+1}^{n} E_{jk} S_{kk} C'_{k\nu} (X_\mu - L_{\mu\mu} X_\mu) + \sum_{k=1}^{n} E_{jk} f_k, \qquad (6.19)$$

$$(j = 1, 2, \ldots, i),$$

$$X_k = \sum_{\nu=1}^{i} S_{kk} C_{k\nu} (Y_\nu - L_{\nu\nu} Y_\nu) + \sum_{\mu=i+1}^{n} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} X_\mu) + f_k \qquad (6.20)$$

$$(k = 1, 2, \ldots, n).$$

Now that the system equation is available, we can proceed with the problem of noninteraction. Several different kinds of noninteraction are considered in /77/. The corresponding matrix representation is given in Figure 6.6.

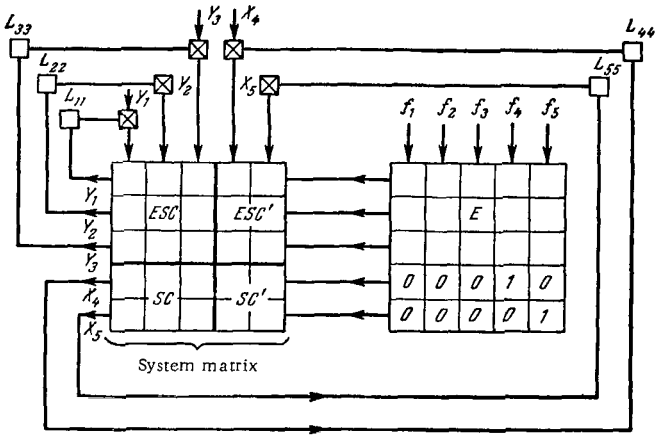

FIGURE 6.6. Illustrating noninteraction conditions.

Noninteraction is first considered in its most elementary sense: the output Y_i is changed only by changing the setting $Y_{i\,\mathrm{ref}}$, and none of the other variations affects this quantity; alternatively by changing the setting $Y_{i\,\mathrm{ref}}$ we change only one controlled variable Y_i, or, in general, each controlled variable is affected only by the variation of its own setting and is independent of other reference values.

Noninteraction is obtained if and only if the system matrix is diagonal. The noninteraction condition is thus that all the nondiagonal elements of the system matrix are zero. For $j = 1, 2, \ldots, i$ and $j \neq t$ we have

$$\sum_{k=1}^{n} E_{jk} S_{kk} C_{kt} = 0 \qquad (6.21)$$

and for all the others, from $\mu = i+1$ to n,

$$S_{\mu\mu} C_{\mu t} = 0 \qquad (6.22)$$

or

$$C_{\mu t} = 0. \tag{6.23}$$

Equations (6.22) and (6.23) show that from $\mu = i+1$ to $\mu = n$, the elements of the t-th column in the matrix C are all zero. The same conclusion obtains from (6.21) for the elements of the column t from $j = 1$ to $j = i$ (when $j \neq t$), i.e.,

$$\sum_{k=1}^{i} E_{jk} S_{kk} C_{kt} = 0. \tag{6.24}$$

Relation (6.24) yields $i-1$ equations in i unknowns, and we may thus write a relationship between any two elements of column t in matrix C.

We now proceed to establish the noninteraction conditions in explicit form. Returning to equations (6.19) and (6.20), we isolated in (6.19) the term with the setting Y_t and write

$$\begin{aligned} Y_j = \sum_{k=1}^{n} \sum_{\substack{\nu=1 \\ \nu \neq t}}^{l} E_{jk} S_{kk} C_{k\nu} (Y_\nu - L_{\nu\nu} Y_\nu) + \\ + \sum_{k=1}^{n} \sum_{\mu=l+1}^{n} E_{jk} S_{kk} C'_{k\nu} (X_\mu - L_{\mu\mu} X_\mu) + \\ + \sum_{k=1}^{n} E_{jk} S_{kk} C_{kt} (Y_t - L_{tt} Y_t) + \sum_{k=1}^{n} E_{jk} f_k \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} X_k = \sum_{\substack{\nu=1 \\ \nu \neq 2}}^{l} S_{kk} C_{k\nu} (Y_\nu - L_{\nu\nu} Y_\nu) + \sum_{\mu=l+1}^{n} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} X_\mu) + f_k + \\ + S_{kk} C_{kt} (Y_t - L_{tt} Y_t). \end{aligned} \tag{6.26}$$

In order for the setting Y_i to influence only the controlled variable Y_j, $j = t$, without interacting with the other outputs, it is necessary that

$$\sum_{k=1}^{l} E_{jk} S_{kk} C_{kt} = 0, \qquad j \neq t. \tag{6.27}$$

Moreover, in (6.26) only the last term may depend on Y_t; all the other terms should vanish, i.e.,

$$S_{kk} C_{k\nu} = 0 \quad \text{for} \quad \nu \neq t, \tag{6.28}$$

or

$$C_{k\nu} = 0,$$

where

$$k = i+1, \ldots, n.$$

Making use of Kronecker's delta

$$\delta_{ik} = 0 \text{ for } i \neq k,$$
$$\delta_{ik} = 1 \text{ for } i = k,$$

we write (6.27) in the form

$$\sum_{k=1}^{i} E_{jk}S_{kk}C_{kt} = \sum_{k=1}^{i} \delta_{jt}E_{jk}S_{kk}C_{kt} \qquad (t = 1, 2, \ldots, i). \tag{6.29}$$

For any given (fixed) t, equation (6.29) gives $i-1$ linear algebraic equations in i unknowns $S_{kk}C_{kt}$, $k = 1, 2, \ldots, n$. Equations (6.29) therefore describe the relationship between these unknowns but do not determine their actual values.

We make use of the following known property of determinants. Putting $|E^*_{jk}|$ for the cofactor of the element E_{jk} in the determinant $|E^*|$ of the square matrix $\|E^*\|$, we may write

$$\sum_{j=1}^{i} E_{jk}|E^*_{jt}| = 0 \qquad \text{for} \quad k \neq t \tag{6.30}$$

and

$$\sum_{j=1}^{i} E_{jk}|E^*_{jt}| = |E^*| \quad \text{for} \quad k = t. \tag{6.30'}$$

Multiplying the two sides of (6.29) by $|E^*_{jt}|$ and summing over j from $j = 1$, to $j = i$, we find

$$\sum_{k=1}^{i} \sum_{j=1}^{i} |E^*_{jt}| E_{jk}S_{kk}C_{kt} = \sum_{k=1}^{i} \sum_{j=1}^{i} |E^*_{jt}| \delta_{it}E_{jk}S_{kk}C_{kt}. \tag{6.31}$$

Making use of (6.30') we now write for $k = l$

$$S_{\rho\rho}C_{\rho t} = \frac{|E^*_{j\rho}|}{|E^*|} \sum_{k=1}^{i} E_{\rho k}S_{kk}C_{k\rho}. \tag{6.32}$$

In particular,

$$S_uC_u = \frac{|E^*_u|}{|E^*|} \sum_{k=1}^{i} E_{tk}S_{kk}C_{kt}. \tag{6.33}$$

Dividing (6.32) through by (6.33), we find

$$\frac{S_{\rho\rho}C_{\rho t}}{S_uC_u} = \frac{|E^*_{j\rho}|}{|E^*_u|} \qquad (\rho, t = 1, 2, \ldots, i). \tag{6.34}$$

We have obtained a relationship in which the nondiagonal matrix elements are expressed in terms of the diagonal elements. Choosing the transfer function SC from (6.29) and (6.34) we ensure the necessary and sufficient conditions of noninteraction. The problem of noninteraction is thus solved for the case when the number of inputs is equal to the number of outputs. In our case, however, the number of inputs is greater than the number of outputs, and we should further consider the choice of C'.

To this end, noninteraction of the variables $\mu = i+1, \ldots, n$ should be ensured. Along the same lines as for y_j, $j = 1, \ldots, i$, it is proved /77/ that the noninteraction conditions for $j = i+1, \ldots, n$ are satisfied if the

transfer functions of the matrix C' are chosen from the following relations:

$$\frac{S_{jj}C'_{jr}}{S_{rr}C'_{rr}} = \frac{-\sum_{k=1}^{i} |E^*_{kj}| E_{kr}}{|E^*|} \qquad (6.35)$$
$$(i=1, \ldots, i,\ r=i+1,\ i+2,\ \ldots,\ n).$$

Summing up, we write the conditions of complete (perfect) noninteraction for any variable in the form

$$C_{\mu t} = 0,$$

where

$$\mu = i+1, \ldots, n,$$
$$t = 1, 2, \ldots, i,$$
$$C'_{\mu t} = 0,$$
$$\mu = i+1, \ldots, n \quad (\mu \neq r),$$
$$r = i+1, \ldots, n,$$
$$\frac{S_{jj}C'_{jt}}{S_{\nu\nu}C_{\nu\nu}} = \frac{|E^*_{ij}|}{|E_{\nu\nu}|},$$

where

$$j,\ t,\ \nu = 1, 2, \ldots, i,$$

and

$$\frac{S_{jj}C_{jr}}{S_{rr}C'_{rr}} = -\frac{\sum_{k=1}^{i} |E^*_{kj}| E_{kr}}{|E^*|},$$
$$(i=1, 2, \ldots, i,\quad r=i+1, \ldots, n).$$

This concludes our discussion of the principal results obtained by Boksenbom and Hood. Further developments by Western authors are mainly based on these results. We will not consider the methods of other authors /81, 82, 78, 79/, since they are of no fundamental interest in connection with the problem at hand. The main conclusion from the preceding discussion of noninteraction conditions which is relevant for our analysis of the problem is that neither the first (Voznesenskii's method) nor the second (Boksenbom and Hood's method) approach discloses the structural features of noninteraction, so that neither is suitable for elucidating the structures in which noninteraction is attainable.

§ 6.3. NONINTERACTION AS A DYNAMIC PROPERTY OF A CERTAIN CLASS OF STRUCTURES

We now consider multivariable control systems with controlled variables interacting through the plant, where the nature of coupling is determined by plant properties. The system comprises n variables, each constituting a closed-loop control subsystem. We shall discuss a number of different cases.

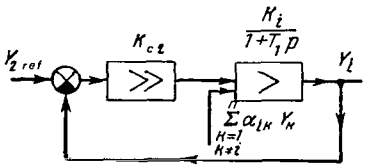

FIGURE 6.7. The i-th control loop with its individual controller.

Case 1. Controlled objects described by first-order differential equations and ideal controllers.

Let each controlled variable be described by a first-order differential equation. The controllers being ideal, their transfer functions are structurally equivalent to gain parameters for each variable. It is clear from § 6.1 that this is the control system investigated by Voznesenskii.

Each single-variable subsystem can be replaced by a structurally equivalent aperiodic loop (see Figure 6.7). The processes in this system are described by the following set of differential equations:

$$\left.\begin{aligned}
&\left[T_1 \frac{d}{dt} + 1 + K_1 K_{c1}\right] Y_1 + K_1 \sum_{i=2}^{n} \alpha_{1i} Y_i = K_1 K_{c1} Y_{1\,ref} + K_1 f_1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\left[T_n \frac{d}{dt} + 1 + K_n K_{cn}\right] Y_n + K_n \sum_{i=1}^{n-1} \alpha_{ni} Y_i = K_n K_{cn} Y_{n\,ref} + K_n f_n,
\end{aligned}\right\} \qquad (6.36)$$

where T_i is the time constant of the plant in relation to the i-th controlled variable, K_i the plant gain in relation to the i-th variable, K_{ci} the controller gain for the i-th variable, α_{ik} coefficient of coupling between i-th and k-th controlled variables. Here α_{ik} is a function of plant properties; it may be a constant or a function described by a differential equation.

Laplace-transforming equations (6.36) and assuming zero initial conditions, we obtain

$$\begin{aligned}
&[T_i p + 1 + K_i K_{c\,i}] Y_i(p) + K_i \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_{ik}(p) Y_k(p) = \\
&\qquad = K_1 K_{c1} Y_{ref}(p) + K_1 f_1(p) \qquad (i = 1, 2, \ldots, n).
\end{aligned} \qquad (6.37)$$

To prevent loss of generality, the coupling coefficient is written as a function of the operator p. Dividing the i-th equation by K_{ci} and putting $\frac{1}{K_{ci}} = m_i$, we find

$$\begin{aligned}
&[m_i (T_i p + 1) + K_i] Y_i(p) + m_i K_i \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik}(p) Y_k(p) = \\
&\qquad = K_i Y_{i\,ref}(p) + m_i K_i f_i(p) \qquad (i = 1, 2, \ldots, n).
\end{aligned} \qquad (6.38)$$

Let us first consider the case $\alpha_{ik}(p) = \alpha_{ik} = \text{const}$. If the controller gains are sufficiently large, i.e., $K_{ci} \to \infty$ and $m_i \to 0$, we see from (6.38) that in the limit the i-th controlled variable depends only on its reference value $Y_{i\,ref}(p)$ and is independent of all other controlled variables. Increasing the gain of each subsystem, we uncouple the various controlled variables in the sense that, to accuracy of $m_i = \frac{1}{K_{ci}}$, the controlled variables no longer interact with one another; we have thus achieved noninteraction to accuracy m_i. This conclusion, however, does not mean much if not supplemented by information on system stability.

If all m_i are of the same order of smallness, the characteristic equation is written as

$$\begin{vmatrix} m(1+T_1p)+K_1 & mK_1\alpha_{12} & \dots & mK_1\alpha_{1n} \\ K_2m\alpha_{21} & m(T_2p+1)+K_2 & \dots & mK_2\alpha_{2n} \\ \dots & \dots & \dots & \dots \\ mK_n\alpha_{n1} & mK_n\alpha_{n2} & \dots & m(1+T_np)+K_n \end{vmatrix} = 0. \qquad (6.39)$$

It is expanded to the form

$$m^nF_n(p)+m^{n-1}F_{n-1}(p)+\dots+F_0(p)=0,$$

where the subscript of F indicates the degree of the polynomial. It follows from Chapter Three that the system is stable as $m \to 0$ if and only if the degenerate equation $F_0(p) = 0$ and the auxiliary equation (of the first kind in this case) each satisfy the respective stability conditions. Here it clearly suffices to test for stability the auxiliary equation only. After simple manipulations it takes the form

$$\prod_{i=1}^{n}(K_i+T_ipm)=0. \qquad (6.40)$$

This is a product of n factors each characterizing an independent damped process for the corresponding controlled variable. The system is thus stable.

The roots of the auxiliary equation are

$$p_i=-\frac{K_i}{mT_i}, \qquad (6.41)$$

which shows that high gain ensures high system stability, i.e., high-speed response, as well as high steady-state accuracy and noninteraction. Moreover, high gain "suppresses" the external disturbance f_i. In this case noninteraction is supplemented by excellent dynamic properties of the system as a whole.

The results also admit of a different interpretation. Suppose that we are interested in improving the dynamic properties of the system. To this end the subsystem gains are increased. Since $F_0(p)$ in this case is a certain constant, the system dynamics at sufficiently small m (large K_{ci}) is completely determined by the properties of the auxiliary equation. It is clear from the expression for the roots of the auxiliary equation (relation (6.41)) that the smaller the parameter m_i the faster is the transient response of the system, i.e., system dynamics is improved by raising the subsystem gains. At the same time uncoupling is achieved and the process is separated into n independent (noninteracting) processes.

Noninteraction is thus derived as a dynamic property of the system at high gain, regardless of whether we are concerned with this particular aspect or not.

Case 2. The plant and the controller are described by first-order differential equations in each controlled variable.

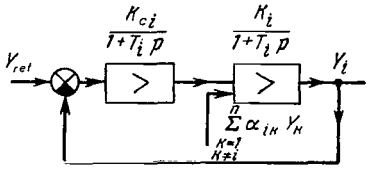

FIGURE 6.8. The i-th control loop with a controller described by a first-order equation.

Figure 6.8 is the structural block diagram of this case. The plant and the controller are represented by aperiodic elements with time constants T_i and T_i' and gains K_i and K_{ci}. Assuming zero initial conditions, we write the following set of equations in Laplace transforms for this system:

$$[(1+T_ip)(1+T_i'p)+K_iK_{ci}]Y_i(p)+K_i(1+T_i'p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=$$
$$=K_iK_{ci}Y_{refi}(p)+K_i(1+T_i'p)f_i(p) \qquad (i=1, 2, \ldots, n). \tag{6.42}$$

Dividing each equation in (6.42) by K_{ci} and putting $\frac{1}{K_{ci}}=m_i$, we obtain

$$[m_i(1+T_ip)(1+T_i'p)+K_i]Y_i(p)+m_iK_i(1+T_i'p)\sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}(p)Y_k(p)=$$
$$=K_iY_{refi}(p)+mK_i(1+T_ip)f_i(p) \qquad (i=1, 2, \ldots, n). \tag{6.43}$$

Here we also take $\alpha_{ik}(p)=\alpha_{ik}(0)$. For $m_i\to 0$ the degenerate part of the set separates into n independent zero-order equations, i.e., structurally the control system is representable by n noninteracting subsystems.

To find the $2n$ roots which recede to infinity as $m_i\to 0$, we have to test the auxiliary equation for stability. If all the m_i are of the same order of smallness, the characteristic equation is finally written in the form

$$m^nF_{2n}(p)+m^{n-1}F_{2n-2}(p)+\ \ldots\ +F_0(p)=0, \tag{6.44}$$

where the subscripts of F indicate the degree of the corresponding polynomial $F(p)$.

It is clear from (6.44) that we will be dealing with an auxiliary equation of the second kind in this case. The auxiliary equation is constructed following the procedure of Chapter Three. After some manipulations, we write it in the form

$$\prod_{i=1}^{n}[(1+T_iq)(1+T_i'q)+K_i]=0. \tag{6.45}$$

Since T_i, T_i', and K_i are always positive real numbers, equation (6.45) satisfies the stability conditions. From (6.45) we also see that the transient response of the system consists of n mutually independent transients corresponding to independent, noninteracting variation of the n controlled variables.

In this more complex case, noninteraction of the individual controlled variables is attained by increasing the subsystem gains.

As in Case 1 we have assumed that K_{ci} are all of the same order of magnitude and can be put equal to one another. This is an inconsequential restriction, since the controller gains can be adjusted accordingly. If, however, the controller gains are different, they are all represented by a single combined gain factor (as in Chapter Three). The rest of the analysis proceeds along the same lines as before, i.e., an auxiliary equation is drawn up, its coefficients incorporating the proportionality coefficients introduced, and is tested for stability.

Without going into the detailed manipulations, we note that the preceding conclusion concerning noninteraction attained by increasing the subsystem gains remain valid in systems where the controllers are structurally representable by integrating elements /39/. For noninteraction, however, there is no need to impose any restrictions on the time constants of the integrating elements (servomotors), as is done, e.g., in /52/.

Case 3. Noninteraction with ideal derivatives.

Consider the general case, when the plant is described by an i-th order differential equation in each controlled variable, and the unstabilized controller by a j-th order differential equation.

High-gain stability is ensured by introducing $n-2$ derivatives ($n=i+j$) into each subsystem according to the rules derived in Chapters Three and Four.

Let $D_i(p)$ be the self-operator of the plant, $M_i(p)$ the self-operator of the controller, K_{ci} the controller gain and K_i the plant gain, all in relation to the i-th controlled variable.

We proceed to derive a set of equations describing the control processes in this system.

The plant equation for the first controlled variable is

$$D_1(p)\,X_1 = K_1\left(X_1' - \sum_{i=2}^{n} a_{1i}X_i + f_1\right). \tag{6.46}$$

The controller equation:

$$M_1(p)\,X_1' = K_{c1}\left[X_{1\,\mathrm{ref}} - X_1 - (a_{10}p^{n_1-2} + a_{11}p^{n_1-3} + \ldots + a_{1n-1}p)\,X_1\right]. \tag{6.47}$$

Inserting for X_1' in (6.46) its expression from (6.47) and proceeding to derive equations for the other controlled variables, we finally obtain

$$\left.\begin{aligned}
&D_1(p)M_1(p) + K_1K_{c1}(a_{10}p^{n_1-2} + a_{11}p^{n_1-3} + \ldots \\
&\qquad \ldots + a_{1,n-1}p) + K_{c1}K_1]\,Y_1 + K_1M_1(p)\sum_{i=2}^{n} a_{1i}Y_i = \\
&\qquad = K_{c1}K_1Y_{1\,\mathrm{ref}} + K_1M_1(p)f_1, \\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&[D_n(p)\,M_n(p) + K_nK_{cn}(a_{n0}p^{n_n-2} + a_{n1}p^{n_n-3} + \ldots \\
&\qquad \ldots + a_{n,n-1}\,p) + K_{ci}K_n]\,Y_n + K_nM_n(p)\sum_{i=1}^{n-1} a_{ni}Y_i = \\
&\qquad = K_{nc}K_nY_{n\,\mathrm{ref}} + K_nM_n(p)f_n,
\end{aligned}\right\} \tag{6.48}$$

where

$$\begin{aligned}
n_1 &= \alpha_1 + \mu_1, \\
&\ldots\ldots \\
n_n &= \alpha_n + \mu_n.
\end{aligned}$$

For sufficiently large K_{ci} the equations in (6.48) degenerate to mutually independent equations. Thus, for $K_{ci}\to\infty$, the i-th equation takes the form

$$K_i\left[a_{i0}p^{n_i-2} + a_{i1}p^{n_i-3} + \ldots + a_{i,n-1}p + K_i\right]Y_i = K_iY_{\mathrm{ref}}. \tag{6.49}$$

The left-hand side of this degenerate equation is a product of factors which constitute the left-hand sides of the degenerate equations of the individual controlled variables.

To establish stability for $m \to 0$, it remains to test the auxiliary equation of second kind. It is also independent of the coupling coefficients α_{ih}.

We thus come to the conclusion that if ideal controllers are used in systems which remain structurally stable at arbitrarily high gain, noninteraction is a dynamic property of the system; the degree of noninteraction increases as the corresponding gain is increased. As regards the quality of control, we see from (6.49) that the degenerate equation, which determines quality, is entirely dependent on the stabilizer parameters. The latter can obviously be chosen so as to ensure required quality.

In what follows we will consider systems where noninteraction cannot be attained by increase of gain alone: the structures should be additionally modified to ensure noninteraction.

§ 6.4. ISOCHRONOUS SYSTEMS

Isochronous systems use an isochronous stabilizer. This is an elastic feedback element having the transfer function

$$\frac{X_{\text{out}}(p)}{X_{\text{in}}(p)} = \frac{\tau p}{1+\tau p}. \tag{6.50}$$

This transfer function is characteristic of mechanical isochronous stabilizers (with negligible piston mass), as well as stabilizing transformers, *RC* elements, and other control elements widely used in practice.

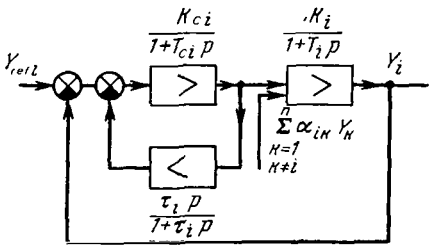

FIGURE 6.9. The i-th control loop with an isochronous controller.

Figure 6.9 is a block diagram of an isochronous system: the elastic feedback loop embraces the controller, which is structurally represented as a single aperiodic element. Using the nomenclature of Figure 6.9, we write

$$\left.\begin{aligned}
&\{(1+T_1p)[(1+T_{c1}p)(1+\tau_1p)+K_{c1}\tau_1p]+K_1K_{c1}(1+\tau_1p)\}Y_1+\\
&\quad+K_1[(1+T_{c1}p)(1+\tau_1p)+K_{c1}\tau_1p]\sum_{i=2}^{n}\alpha_{1i}Y_i=\\
&\quad=K_{c1}K_1(1+\tau_1p)Y_{1\text{ref}}+K_1[(1+\tau_1p)(1+T_{c1}p)+K_{c1}\tau_1p]f_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\{(1+T_np)[(1+T_{c1}p)(1+\tau_np)+K_{cn}\tau_np]+K_nK_{cn}(1+\tau_np)\}Y_n+\\
&\quad+K_n[(1+T_{cn}p)(1+\tau_np)+K_{cn}\tau_np]\sum_{i=1}^{n-1}\alpha_{ni}Y_i=\\
&\quad=K_{cn}K_n(1+\tau_np)Y_{n\text{ref}}+K_n[(1+\tau_np)(1+T_{cn}p)+K_{cn}\tau_np]f_n.
\end{aligned}\right\} \tag{6.51}$$

It is clear from Figure 6.9 that the subsystem gains can be increased indefinitely by increasing the corresponding controller gains.

The system as a whole is stable if the degenerate equation and the auxiliary equation each satisfy the stability conditions.

To explore the possibility of noninteraction, we divide all the equations in (6.51) by K_{ci}. Putting $\frac{''1}{K_{ci}} = m_i$ and assuming all m_i to be of the same order of smallness, we draw up an auxiliary equation (it turns out to be an equation of the first kind in this case).

After simple manipulations, we obtain the auxiliary equation in the form

$$\prod_{i=1}^{n} (1 + T_{c\,i} q) = 0. \tag{6.52}$$

It always satisfies the stability criteria and is independent of the coupling coefficients.

Let us now consider the degenerate equations. Dividing (6.51) by K_{ci} and putting $m = \frac{1}{K_{ci}} = 0$, we obtain the following set of degenerate equations

$$\left.\begin{aligned}
&[(1 + T_1 p)\,\tau_1 p + K_1 (1 + \tau_1 p)]\, Y_1 + K_1 \tau_1 p \sum_{i=2}^{n} \alpha_{1i} Y_i = \\
&\qquad = K_1 (1 + \tau_1 p)\, Y_{1\,\mathrm{ref}} + K_1 \tau_1 p f_1, \\
&[(1 + T_2 p)\,\tau_2 p + K_2 (1 + \tau_2 p)]\, Y_2 + K_2 \tau_2 p \sum_{i=1}^{n} \alpha_{2i} Y_i = \\
&\qquad = K_2 (1 + \tau_2 p)\, Y_{2\,\mathrm{ref}} + K_2 \tau_2 p f_2 \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&[(1 + T_n p)\,\tau_n p + K_n (1 + \tau_n p)]\, Y_n + K_n \tau_n p \sum_{i=1}^{n-1} \alpha_{ni} Y_i = \\
&\qquad = K_n (1 + \tau_n p)\, Y_{n\,\mathrm{ref}} + K_n \tau_n p f_n.
\end{aligned}\right\} \tag{6.53}$$

Each equation in (6.53) contains terms which account for the interaction of the various controlled variables. Noninteraction thus cannot be ensured by simple increase of the gain alone.

To ensure noninteraction, the structure is modified as follows. The sum of all variables, with the exception of the variable corresponding to the particular controller, is additionally delivered to the input of the isochronous stabilizer of each controller. The system behavior is thus described by the following equations.

First-variable plant equation (see Figure 6.9):

$$(1 + T_1 p)\, Y_1 = K_1 \left[Y_1' + \sum_{i=2}^{n} \alpha_{1i} Y + f_1 \right]. \tag{6.54}$$

Controller equation

$$(1 + T_{c1} p) Y_1' = K_{c1} \left[Y_{1\,\mathrm{ref}} - Y_1 - \frac{\tau_1 p}{1 + \tau_1 p} Y_1' - \sum_{i=2}^{n} \frac{\tau p}{1 + \tau_1 p} \alpha_{1i} Y_i \right]. \tag{6.55}$$

Inserting for Y_1' in (6.54) its expression from (6.55) we find

$$\{(1+T_1p)[(1+T_{c1}p)(1+\tau_1p)+K_{c1}\tau_1p]+K_1K_{c1}(1+\tau_1p)\}Y_1-$$
$$-K_1[(1+T_{c1}p)(1+\tau_1p)+K_{c1}\tau_1p]\sum_{i=2}^{n}\alpha_{1i}Y_i+K_1K_{c1}\tau_1p\sum_{i=2}^{n}\alpha_{1i}Y_i=$$
$$=K_1K_{c1}(1+\tau_1p)Y_{1\,\mathrm{ref}}+K_1f_1[(1+T_{c1}p)(1+\tau_1p)+K_{c1}\tau_1p]. \qquad (6.56)$$

Similar equations are obtained for the other variables. For the k-th variable we thus have

$$\{(1+T_kp)[(1+T_{ck}p)(1+\tau_kp)+K_{ck}\tau_kp]+K_kK_{ck}(1+\tau_kp)\}Y_k-$$
$$-K_k[(1+T_{ck}p)(1+\tau_kp)+K_{ck}\tau_kp]\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_{ki}Y_i+$$
$$+K_kK_{ck}\tau_kp\sum_{i=1}^{n}\alpha_{ki}Y_i=K_kK_{ck}(1+\tau_kp)Y_{k\,\mathrm{ref}}+$$
$$+K_kf_k[(1+T_{ck}p)(1+\tau_kp)+K_{ck}\tau_kp]. \qquad (6.57)$$

Dividing each equation in (6.57) by the appropriate K_{ci} and taking $m_i=\frac{1}{K_{ci}}$ to be sufficiently small, we obtain a set of independent degenerate equations for each controlled variable. Thus, for the k-th controlled variable,

$$[(1+T_kp)\tau_kp+K_k(1+\tau_kp)]Y_k=K_k(1+\tau_kp)Y_{k\,\mathrm{ref}}+K_k\tau_kpf_k. \qquad (6.58)$$

This equation describes an independent, noninteracting process in the k-th control loop. This process is independent of the other controlled variables.

Comparison with the results of the previous sections shows that, whereas in the preceding structures increase of gain ensured noninteraction and simultaneous rejection of external disturbances, noninteraction in a system representable by equation (6.58) is not accompanied by disturbance rejection. This important property will be investigated in what follows during a detailed analysis of noninteraction and invariance.

§ 6.5. NONINTERACTION IN THE GENERAL CASE

Let us consider noninteraction in the general case of a system with n interacting variables which remains stable as the individual gains are increased indefinitely. We assume that each control loop is stabilized by a device with a transfer function $\frac{F_{ni}(p)}{F_{mi}(p)}$ which encloses part of the controller with the self-operator $M_{i\,\mathrm{st}}$ (Figure 6.10).

Let $D_i(p)$ be the self-operator of the plant in respect to the i-th controlled variable, $M_{i\,\mathrm{st}}(p)$ the self-operator of the stabilized section, $M_{i\,\mathrm{deg}}(p)$ the self-operator of the unstabilized section, $K_{i\,\mathrm{deg}}$ the gain of the unstabilized section.

It is easily seen that in this case, as in § 6.3, gain alone does not ensure noninteraction. We do not prove this proposition, as it partly follows from the results of § 6.3, where the stabilizer transfer function is a particular case of the function $\frac{F_n(p)}{F_m(p)}$.

Let a sum of disturbances $\sum_{k=1,2,\ldots,i,\ldots,n} \alpha_{ik} Y_k$ be additionally delivered to the input of each stabilizer. The system is thus described by

$$\begin{aligned}
&\{D_i(p) M_{i\,\deg}(p)[M_{i\,st}(p) F_{mi}(p) + K_{i\,st} F_{ni}(p)] + \\
&\quad + K_{i\,st} K_{i\,\deg} K_i F_{mi}(p)\} Y_i = K_i K_{i\,st} K_{i\,\deg} F_{mi}(p) Y_{i\,ref} - \\
&\quad - K_i M_{i\,\deg} K_{i\,st} F_{ni}(p) \sum_{k=1}^{n} \alpha_{ik} Y_k + \\
&\quad + K_i M_{i\,\deg}(p)[M_{i\,st}(p) F_{mi}(p) + K_{i\,st} F_{ni}(p)] \sum_{k=1}^{n} \alpha_{ik} Y_k + \\
&\quad + K_i M_{i\,\deg}(p)[M_{i\,st}(p) F_{mi}(p) + K_{i\,st} F_{ni}(p)] f_1,
\end{aligned} \tag{6.59}$$

where $n = 1, 2, \ldots, n$.

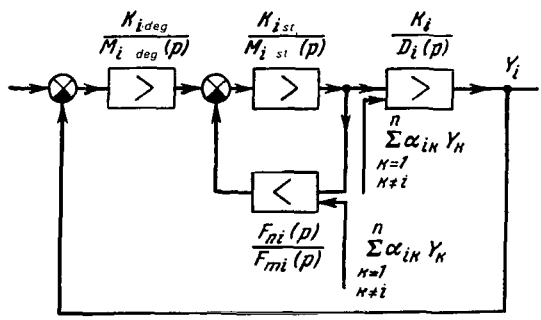

FIGURE 6.10. Illustrating the derivation of noninteraction in the general case of a lagless plant.

As we have already indicated, the stabilizer transfer function and the stabilizer connections are chosen so that the system remains stable despite indefinite increase in gain. $K_{i\,st}$ may therefore be increased indefinitely in each subsystem. The entire system will be stable if the degenerate and the auxiliary equation each satisfy the corresponding stability criteria.

Let $K_{i\,st}$ be sufficiently large. Dividing each equation in (6.59) by $K_{i\,st}$ we find that for sufficiently small $m_i = \frac{1}{K_{i\,st}}$ the system separates into n independent equations, each describing one controlled variable only.

The stabilizer input in this case is thus the sum of all extraneous controlled variables, which implies noninteraction for $K_{st} \to \infty$.

It is clear from the preceding that if the noninteraction conditions are satisfied, each subsystem can be selected and investigated independently with the aid of the well-known methods of the theory of single-variable control systems.

It should be emphasized that the accuracy of the results is conditioned by the order of smallness of m. The smaller the parameter m, i.e., the higher the gain K_{st}, the more accurate are the results.

In practice, if all the other system parameters are known, $K_{i\,st}$ is readily chosen by the methods described in /39/.

The analysis of this section leads to the following conclusions.

(a) Multivariable control systems which retain their stability at any (including arbitrarily large) gain can be reduced to noninteracting systems. In the general case noninteraction is attained by increasing the corresponding gain $K_{i\,\text{п}}$ and simultaneously delivering to the stabilizer input the sum of disturbances from all the extraneous variables with proportionality coefficients α_{ik}.

(b) The methods of the theory of single-variable control systems are fully applicable to the design of noninteracting multivariable systems and to the choice of system parameters.

§ 6.6. NONINTERACTION IN SYSTEMS WITH LAGS

In our discussion of stability of lagged systems the input variable of the time-lag element was assumed to be zero during the time from $t = -\tau$ to $t = 0$. This assumption is valid, since in linear systems (with which we are concerned) stability is independent of the initial conditions. However, in the general case, zero initial conditions are inadmissible in systems with lags. The system equations should therefore be written for nonzero initial conditions.

We consider the following cases: (a) high-gain stability is attained by introducing ideal derivatives into the systems, and (b) high-gain stability is attained by introducing real derivatives.

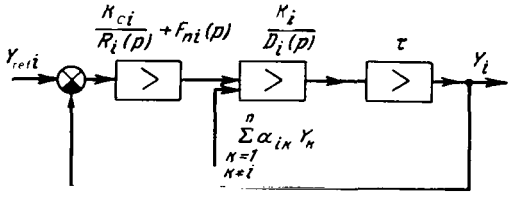

FIGURE 6.11. Illustrating noninteraction in the general case of a lagged plant.

Let us derive the equation of the i-th subsystem for case (a). Figure 6.11 is a block diagram of this case.

The plant equation ignoring time lag is

$$D_i(p)\,Y_i = K_i[x_i + \sum \alpha_{ik} Y_k + f_i]. \tag{6.60}$$

The equation of the lag element

$$Y_i' = Y_i(t - \tau). \tag{6.61}$$

The controller equation

$$R_i(p)\,x_i = K_{ci}[Y_{ref\,i} - Y_i'(a_0 p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-3}p + 1)], \tag{6.62}$$

where

$$a_0 p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-3} p = F(p),$$

and $n = D + R$ is the overall order of the differential equation describing processes in this subsystem.

Laplace-transforming equations (6.60), (6.61), and (6.62), we obtain the plant equation

$$D_i(p) Y_i = K_i' [x_i(p) + \sum \alpha_{ik} Y_k(p) + f_i(p)] + D_{in}[P, x_i(0), Y_i'(0)], \qquad (6.63)$$

the lag equation

$$Y_i'(p) = e^{-\tau_i p} Y_i(p) + \int_{-\tau_i}^{0} Y_i' e^{-pt}\, dt = e^{-\tau_i p} Y_i(p) + \psi(p), \qquad (6.64)$$

where $\psi_i(p) = \int_{-\tau}^{0} x_i' e^{-pt} dt$ accounts for the initial conditions during the time from $t = -\tau$ to $t = 0$,

and the controller equation

$$R_i(p) x_i(p) = K_{ci} [Y_{ref\,i} - Y_i(p)(a_0 p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-3} p + 1)] + R_{ni}[P, Y_i(0), x_i'(0)]. \qquad (6.65)$$

In (6.64) and (6.65), D_{ni} and R_{ni} are contributed by the initial conditions for Y_i and X_i, respectively. Solving equations (6.63), (6.64), and (6.65) for the Laplace-transform of the controlled parameter, $Y_i(p)$, we find

$$\begin{aligned} &[D_i(p) R_i(p) e^{\tau p} + K_i K_{ci}(a_0 p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-3} p + 1)] Y_i(p) + \\ &+ K_i R_i(p) \sum \alpha_{ik} Y_k(p) e^{\tau p} = \{K_i K_{ci} Y_{i\,ref} + K_i \psi(p)(a_0 p^{n-2} + \\ &+ a_1 p^{n-3} + \ldots + 1) + K_i R_{ni}[P, Y_i(0), Y_i'(0) \ldots] + \\ &+ K_i R_i(p) f_i(p) + R_i(p) D_n[P, Y_i(0), Y_i'(0) \ldots]\} e^{\tau p}. \end{aligned} \qquad (6.66)$$

Dividing the two sides of (6.66) by K_{ci} and taking K_{ci} to be sufficiently large, we find

$$\begin{aligned} &m D_i(p) R_i(p) e^{\tau p} + K_i(a_0 p^{n-2} + a_1 p^{n-3} + \ldots + 1) Y_i(p) + \\ &+ m K_i R_i(p) \sum \alpha_{ik} Y_k(p) e^{\tau p} = \{K_i Y_{i\,ref} + m K_i \psi(a_0 p^{n-2} + \\ &+ a_1 p^{n-3} + \ldots + 1) + m K_i R_{ni}[p x_i(0), x_i'(0) \ldots] + \\ &+ m K_i R_i(p) f_i(p) + m R_i(p) D_{nn}[P, Y_i(0), Y_i'(0) \ldots]\} e^{\tau p}. \end{aligned} \qquad (6.67)$$

First let us consider the effect of nonzero initial conditions. We assume that during the time from $-\tau$ to 0 the function Y_i and its derivatives have finite values; we moreover assume that at the time $t = 0$ the function $x(0)$ and its derivatives are also finite. Equation (6.67) then shows that as K_{ci} increases, the effect on the transient response of the initial values of Y and x and of their derivatives diminishes, dropping to zero in the limit. However, a distinctive feature of this stabilization technique (introduction of ideal derivatives to the system input), as is seen from (6.67), is that

in the limit, when $m \to 0$, the equation becomes lagless, the leading term vanishes, and instability sets in. Introduction of ideal derivatives to the system input is thus inadmissible for purposes of attaining high-gain noninteraction.

This conclusion is fairly obvious without a formal proof. Introduction of ideal derivatives to the system input is equivalent to stabilizing the entire system with a stabilizer

$$a_0 p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-3} p.$$

This system is unstable as $K_{cl} \to \infty$, since the stabilizer embraces the lag element as well.

It is significant, however, that the initial conditions, as long as they are finite, do not affect the fundamental properties of the system. In what follows we therefore omit the particular initial conditions.

Let us now consider the case when the ideal derivative is so introduced that the system remains stable in the limit despite an indefinite increase in the gain of the stabilized section.

For the i-th subsystem we have

$$\begin{aligned}
&[D_i(p) N_i p) Q_i(p) e^{\tau_i p} + K_{i\,st}\, D_i(p) N_i(p) F_{ni}(p) e^{\tau_i p} + K_{i\,tot}]\, Y_i + \\
&\quad + K_i \{N_i(p) Q_i(p) + K_{i\,st} N_i(p) F_{ni}(p)\} \sum \alpha_{ik} Y_k = \\
&\quad = [N_i(p) Q_i(p) + K_i K_{i\,st} N_i(p) F_{ni}(p)]\, Y_{i\,ref} + \\
&\qquad + [N_i(p) Q_i(p) + K_{i\,st} N_i(p) F_{ni}(p)]\, f_i,
\end{aligned} \tag{6.68}$$

where $K_{i\,tot} = K_i K_{Ni} K_{i\,st}$.

The order of the highest ideal derivative entering the expression for $F_{ni}(p)$ is chosen so as to ensure stability with arbitrarily large $K_{i\,st}$.

We divide (6.68) by $K_{i\,st}$ and put $K_{i\,st} \to \infty$. The degenerate equation takes the form

$$\begin{aligned}
&[D_i(p) N_i(p) F_{ni}(p) e^{\tau_i p} + K_{i\,tot}]\, Y_i + \\
&\quad + K_i N_i(p) F_{ni}(p) \sum \alpha_{ik} Y_k = K_{i\,tot}\, Y_{i\,ref} + N_i(p) F_{ni}(p) f_i.
\end{aligned} \tag{6.69}$$

It is clear from (6.69) that noninteraction cannot be ensured simply by increasing the gain $K_{i\,st}$, since the degenerate equation contains a term

$$N_i(p) F_{ni}(p) \sum_{k=1,\,2,\,\ldots,\,i-1,\,i+1,\,n} \alpha_{ik} Y_k,$$

and the process is consequently dependent on the contribution from all the other controlled variables.

Noninteraction is attained by proceeding along the same lines as with lagless systems (6.2). Signals of the form

$$\sum_{k=1,\,2,\,\ldots,\,n} \alpha_{ik} Y_k \tag{6.70}$$

are delivered to the stabilizer inputs. The set of equations for the first subsystem is written as

$$[D_1(p)N_1(p)Q_1(p)e^{\tau_1 p}+K_{1\,st}D_1(p)N_1(p)F_1(p)e^{\tau_1 p}+K_{1\,tot}]Y_1+$$
$$+[N_1(p)Q_1(p)+K_{1\,st}N_1(p)F_{n1}(p)]\sum_{i=2}^{n}\alpha_{1k}Y_k-$$
$$-K_{1\,st}N_1(p)F_{n1}(p)\sum_{i=2}^{n}\alpha_{1k}Y_k=$$
$$=K_1'[N_1(p)Q_1(p)+K_{1\,st}N_1(p)F_{n1}(p)]Y_{1\,ref}+$$
$$+[N_1(p)Q_1(p)+K_{1\,st}N_1(p)F_{n1}(p)]f_1. \qquad (6.71)$$

Similar expressions are obtained for the other subsystems: it is only necessary to substitute the appropriate subscript for 1 and to omit it from the sum $\sum_{i=2}^{n}$.

Dividing (6.71) by $K_{1\,st}$ and putting $K_{1\,st}\to\infty$, we obtain the degenerate equation

$$[D_1(p)N_1(p)F_{n1}(p)e^{-\tau_1 p}+K_{1\,deg}]Y_1=$$
$$=K_1'N_1(p)F_{n1}(p)Y_{1\,ref}+N_1(p)F_{n1}(p)f_1. \qquad (6.72)$$

This equation is independent of $\alpha_{ik}x_k$.

We thus conclude that introduction of an additional signal (6.70) to the stabilizer inputs in conjunction with an increase in the gain $K_{i\,st}$ ensures noninteraction of the individual subsystems in a multivariable control system with lag elements. It is of course implied that the conditions of infinite-gain stability are satisfied.

In conclusion let us consider a case when stability at infinite gain $K_{i\,st}$ is ensured by real stabilizers in the system.

The transfer function of the stabilizer in the i-th subsystem is

$$\frac{F_{ni}(p)}{F_{mi}(p)}.$$

Acting along the same lines as in the preceding we readily show that noninteraction can be attained by indefinitely increasing the gain of the stabilized section in the i-th subsystem if an additional signal $\sum_{k=1}^{n}\alpha_{ik}Y_k$ is delivered to the stabilizer input (the subscript k takes on all the values except i). The noninteraction criteria of the previous sections are thus extended to systems with lag elements as well. Note that noninteraction does not ensure automatic rejection of external disturbances in this case either.

§ 6.7. INVARIANCE PRINCIPLE

In invariant control systems the generalized coordinate of the system, in particular the controlled variable, is independent of the external disturbances. They are therefore also called systems with rejection of external disturbances.

In multivariable control systems the problem of invariance is substantially complicated by interaction between the various controlled variables, which is superimposed on the external disturbances. There is, however, a fundamental difference between the effect of external disturbances and the effect of coupling in the system: external disturbances do not affect the stability of the system as a whole, whereas the coupling coefficients, or in the general case the coupling operators, have a substantial influence on system stability.

To elucidate the problem of invariance, we first consider a three-variable system and then extend the results to the general case.

1. A three-variable system

Consider a system with controlled variables interacting through the plant. Analysis of combined control systems has shown that the external disturbance, if appropriately channeled into the system, can make the controlled variable independent of external disturbances. The feasibility of this rejection procedure follows, say, from the Poncelet principle of load control. It is therefore interesting to consider this problem in application to control systems operating on the Watt—Polzunov principle (control by deviation).

The equation for the controlled variable Y_1 in this case can be derived from (5.22) putting

$$\beta_{ik}=0, \quad \Theta_{ni}=0 \text{ and } \Theta_{mi}=1 \quad (i\neq k;\ i=1,\ 2,\ 3). \tag{6.73}$$

Making use of (6.73), we obtain from (5.22)

$$Y_1=\frac{1}{\Delta_3}\sum_{k=1}^{3}(-1)^{1+k}A_{1k}K_{c\,k}l_{kk}Y_{rcf\,k}+\frac{1}{\Delta_3}\sum_{i=1}^{3}(-1)^{1+i}A_{1i}\left[K_{c\,i}d_{ii}+N_{ii}\right]f_i, \tag{6.74}$$

where

$$\Delta_3=\begin{vmatrix} a_{11}+K_{c\,1}b_{11} & (c_{11}+K_{c\,1}d_{11})\alpha_{12} & (c_{11}+K_{c\,1}d_{11})\alpha_{13} \\ (c_{22}+K_{c\,2}d_{22})\alpha_{21} & a_{22}+K_{c\,2}b_{22} & (c_{22}+K_{c\,2}d_{22})\alpha_{23} \\ (c_{33}+K_{c\,3}d_{33})\alpha_{31} & (c_{33}+K_{c\,3}d_{33})\alpha_{32} & a_{33}K_{c\,3}b_{33} \end{vmatrix}, \tag{6.75}$$

$$\left.\begin{aligned} A_{11}&=\begin{vmatrix} a_{22}+K_{c\,2}b_{22} & (c_{33}+K_{c\,3}d_{33})\alpha_{32} \\ (c_{22}+K_{c\,2}d_{22})\alpha_{23} & a_{33}+K_{c\,3}b_{33} \end{vmatrix}, \\ A_{12}&=\begin{vmatrix} (c_{11}+K_{c\,1}d_{11})\alpha_{12} & (c_{33}+K_{c\,3}d_{33})\alpha_{32} \\ (c_{11}+K_{c\,1}d_{11})\alpha_{13} & a_{33}+K_{c\,3}b_{33} \end{vmatrix}, \\ A_{13}&=\begin{vmatrix} (c_{11}+K_{c\,1}d_{11})\alpha_{12} & a_{22}+K_{c\,2}b_{22} \\ (c_{11}+K_{c\,1}d_{11})\alpha_{13} & (c_{22}+K_{c\,2}d_{22})\alpha_{23} \end{vmatrix}. \end{aligned}\right\} \tag{6.76}$$

Here

$$\left.\begin{aligned} a_{ii}&=Q_iR_iF_{mi}D_i, \\ b_{ii}&=M_iK_iF_{mi}+D_iQ_iF_{ni}, \\ d_{ii}&=K_iQ_iF_{ni}, \\ l_{ii}&=K_iM_iF_{mi}, \\ N_{ii}&=K_iQ_iR_iF_{mi}. \end{aligned}\right\} \tag{6.77}$$

To reject all the external disturbances, the coefficient of f_i should be made equal to zero, i.e.,

$$(-1)^{i+1} A_{1k} [K_{ci} d_{ii} + N_{ii}] = 0 \quad \text{for} \quad i = 1, 2, 3.$$

This coefficient is made up of two factors:

$$A_{ik} \quad \text{and} \quad [K_{ci} d_{ii} + N_{ii}] = 0. \tag{6.78}$$

Consider the first factor, A_{ik}. It is clear from (6.74) that the condition $A_{ik} = 0$ is equivalent to rejection of the entire control system, since elimination of the external disturbances simultaneously eliminates all the reference values Y_{wfk} and we no longer have a control system. Furthermore, if all A_{ik} are zero, the denominator vanishes and we arrive at the identity $0 \equiv 0$.

Here lies the main error of G.V. Shchipanov who proposed control by deviation for the realization of the fundamentally sound idea of disturbance rejection that he had developed. It is not only that a zero identity is obtained: a more significant fallacy is that by rejecting the external disturbance in this way we simultaneously eliminate the reference values and thus destroy our control system. Moreover, in deviation control systems, on passing to the limit $A_{ik} = 0$, we must investigate the stability under arbitrarily small deviation from equality, i.e., test the system for coarseness in the sense of A.A. Andronov.

Now consider the factor $(K_{ci} d_{ii} + N_{ii})$. Inserting for d_{ii} and N_{ii} their expressions from (6.77), we find

$$K_{ci} d_{ii} + N_{ii} = K_i Q_i [K_{ci} F_{ni} + R_i F_{mi}] = 0,$$

or

$$\frac{K_{ci}}{R_i} = -\frac{F_{mi}}{F_{ni}}. \tag{6.79}$$

Let us elucidate the physical meaning of condition (6.79). It is easily seen that (6.79) calls for the introduction of ideal derivatives. Indeed, K_{ci} is a constant positive quantity, R_i is the controller self-operator. F_{mi}, the self-operator of the stabilizer, is thus clearly a constant number, whereas the stabilizer numerator F_{ni} must be precisely equal to the controller self-operator $R_i(p)$. We conclude that in this case control by deviation in principle cannot ensure complete (perfect) rejection of external disturbances. Disturbance rejection is possible only to an accuracy of some ε, and the equation should be investigated for coarseness.

Various techniques ensuring invariance have been proposed in the theory and practice of automatic control /14, 26, 27, 29, 51, 56/. We do not intend to discuss each and every of these methods; only the most typical cases will be considered, with particular reference to their advantages and, possibly, shortcomings. Note that condition (6.79) is not the only one that ensures invariance; moreover, this is not the best policy for obtaining ε-invariance of a control system.

The point is, that a stabilizer is incorporated in the system to ensure stability and provide certain dynamic properties. If the stabilizer is

chosen on the basis of condition (6.79), it is by no means clear that it will meet the required performance characteristics as far as system dynamics is concerned. A better solution is to tackle the problem of ε-variance simultaneously with the problem of stability and system performance.

We have considered the particular case of a three-variable system, but the results are obviously valid in the general case of n controlled variables too.

2. Application of local positive feedback to ensure invariance

We will now consider some instances of invariance ensured by local positive feedback. Our interest in these methods stems not from their practical significance but from the fact that they provide an excellent opportunity to warn the reader against various fallacies and erroneous conclusions that may lead to undesirable results.

Figure 6.12 is a block diagram of the system analyzed in /29/. Using the nomenclature of Figure 6.12, we write

$$\left.\begin{aligned} y &= \frac{1}{a_{22}}(u_{\text{ref}} - x + v), \\ z &= \frac{a_{32}}{a_{33}} y, \\ v &= z a_{23}, \\ x &= (a_{13} y + f(p)) \frac{1}{a_{11}}. \end{aligned}\right\} \tag{6.80}$$

Solving (6.80) for x, we obtain

$$a_{11}a_{13}(a_{23}a_{32} - a_{22}a_{23}) + a_{13}a_{33}]\, x = a_{13}a_{33}u_{\text{ref}} + (a_{23}a_{32} - a_{22}a_{33})\, f(p). \tag{6.81}$$

This is a single-variable control system, and the invariance conditions can be determined without considering the general case of a multivariable system. The particular results, as always, can be easily generalized to the case of n controlled variables.

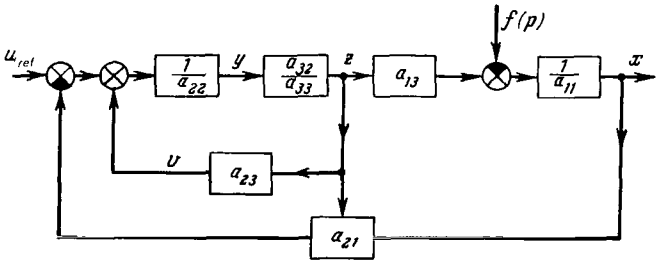

FIGURE 6.12. Illustrating the conditions of disturbance rejection.

We see from (6.81) that the controlled variable x is independent of external disturbances if

$$a_{23}a_{32} - a_{22}a_{33} \equiv 0. \tag{6.82}$$

Let the parameters be chosen so that identity (6.82) is satisfied. It is readily seen that the system is noncoarse for perfect invariance, whereas for ε-invariance it is realizable only if certain additional conditions are imposed. Indeed, the degree of the polynomial $a_{11}a_{13}(a_{23}a_{32}-a_{22}a_{33})$ is always higher than the degree of the polynomial $a_{13}a_{33}$. Under these conditions, an arbitrarily small departure from identity gives rise to roots which recede to infinity; the number of these roots is determined by the difference in the degrees of the polynomials

$$a_{11}a_{13}(a_{23}a_{32}-a_{22}a_{33}) \quad \text{and} \quad a_{13}a_{33}.$$

If the new terms introduced by departure from identity have a minus sign, the system is unstable irrespective of this difference. Perfect invariance is thus unfeasible, since, in view of the possible appearance of terms with negative coefficients, the system is not coarse in the sense of A.A. Andronov.

Let us now consider the case of ε-invariance. The question can be discussed at all only if

$$a_{23}a_{32}-a_{22}a_{33}>0. \tag{6.83}$$

However, if (6.83) is satisfied and also

$$a_{23}a_{32}-a_{22}a_{33}\leqslant\varepsilon,$$

the system is realizable only if the difference in the degrees of $a_{11}a_{13}(a_{23}a_{32}-a_{22}a_{33})$ and $a_{13}a_{33}$ is not greater than two and the degenerate equation

$$a_{13}a_{33}=0 \tag{6.84}$$

meets the stability criteria. Furthermore, the additional requirements described in Chapter Three should also be satisfied. In other words, the realizability problem reduces to the design of a structure which remains stable at infinite gain.

3. Invariance via feedback

Let us consider another method, which ensures invariance with the aid of internal feedback /20/.

Figure 6.13 is a block diagram of a control system, where invariance in relation to the disturbance $e(t)$ is attained by introducing local feedback with appropriately chosen transfer functions. The exact character of the local feedback will be decided at a later stage. Meanwhile, taking all the feedbacks with the plus sign and using the nomenclature of Figure 6.13, we obtain the following set of operator equations:

$$U_1(p)=Z_1(p)\,X(p)+Z_2(p)\,U_4(p)+Z_3(p)\,U_2(p)+Q(p), \tag{6.85}$$
$$U_2(p)=W_1(p)\,U_1(p), \tag{6.86}$$
$$U_3(p)=W_2(p)\,U_2(p), \tag{6.87}$$
$$U_4(p)=U_3(p)+e(p), \tag{6.88}$$
$$X(p)=W_3(p)\,U_4(p), \tag{6.89}$$

where $X(p)$ is the Laplace transform of the controlled variable, $e(p)$ the transform of the disturbance, W_1, W_2, W_3, Z_2, and Z_3 are the transfer functions of the system elements and the local feedbacks. The system is designed according to the Watt—Polzunov principle, and so $Z_1 = -1$.

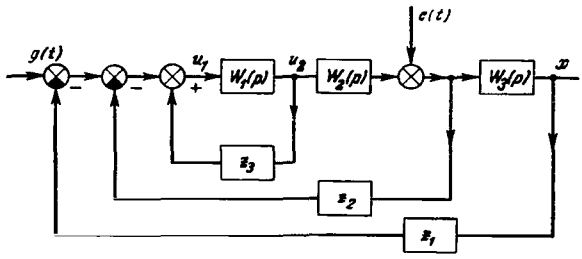

FIGURE 6.13. Illustrating the derivation of disturbance rejection conditions with the aid of positive feedback.

Solving equations (6.85)—(6.89) for the transform of the controlled variable $X(p)$, we find

$$X(p) = \frac{W_1(p)\,W_2(p)\,W_3(p)\,Q(p) + W_3(p)\,[1 - W_1(p)\,Z_3(p)]\,e(p)}{[1 - W_1(p)\,Z_3(p)] + W_1(p)\,W_2(p)\,W_3(p) - W_1(p)\,W_2(p)\,Z_2(p)} \tag{6.90}$$

In order for the controlled variable $x(t)$ to be independent of the disturbance $e(t)$, the coefficient of $e(p)$ in (6.90) should be equal to zero, i.e.,

$$1 - W_1(p)\,Z_3(p) = 0, \tag{6.91}$$

or

$$Z_3(p) = \frac{1}{W_1(p)}. \tag{6.92}$$

If the transfer functions are represented as rational-fractional functions of the operator p, then putting

$$Z_3(p) = \frac{F_{n3}(p)}{F_{m3}(p)} \quad \text{and} \quad W_1(p) = \frac{K_1(p)}{D_1(p)}, \tag{6.93}$$

we find

$$\frac{F_{n3}(p)}{F_{m3}(p)} = \frac{D_1(p)}{K_1(p)}. \tag{6.94}$$

Note that a similar condition was derived in the preceding subsection, where invariance was achieved via local positive feedback. The feedback with the transfer function Z_3 should apparently be positive in this case, too; moreover, for the real components which make up the basic elements of control systems, the degree of the denominator $D_1(p)$ is greater than the degree of the numerator $K_1(p)$. In most cases, $K_1(p)$ is simply the gain, i.e., a constant positive number. We thus again arrive at the requirement

of local positive feedback with ideal derivatives. In other words, nothing new has been hitherto derived in addition to what we considered in § 6.4.

Let us now consider the stability of this system and test it for coarseness in the sense of Andronov. We put for the transfer functions

$$\left.\begin{aligned} W_1(p)&=\frac{K_1(p)}{D_1(p)}, \quad W_2(p)=\frac{K_2(p)}{D_2(p)}, \\ W_3(p)&=\frac{K_3(p)}{D_3(p)}, \quad Z_2(p)=\frac{F_{n2}(p)}{F_{m2}(p)}. \end{aligned}\right\} \tag{6.95}$$

The characteristic equation needed for stability analysis is obtained by putting the denominator in the right-hand side of (6.90) equal to zero. Making use of (6.95), we write for the characteristic equation

$$\left[1-\frac{F_{n3}(p)}{F_{m3}(p)}\frac{K_1(p)}{D_1(p)}\right]+\frac{K_1(p)K_2(p)K_3(p)}{D_1(p)D_2(p)D_3(p)}-\frac{K_1(p)K_2(p)}{D_1(p)D_2(p)}\frac{F_{n2}(p)}{F_{m2}(p)}=0$$

or

$$\begin{aligned} &D_1(p)D_2(p)D_3(p)F_{m2}(p)\left[1-\frac{F_{n3}(p)}{F_{m3}(p)}\frac{K_1(p)}{D_1(p)}\right]+ \\ &+K_1(p)K_2(p)K_3(p)F_{m2}(p)-K_1(p)K_2(p)D_3(p)F_{n2}(p)=0. \end{aligned} \tag{6.96}$$

It is clear that the system degenerates whenever condition (6.94) is satisfied, i.e., if

$$1-\frac{F_{n3}(p)}{F_{m3}(p)}\frac{K_1(p)}{D_1(p)}=0,$$

since the degree of the polynomial before the expression in brackets

$$1-\frac{F_{n3}(p)}{F_{m3}(p)}\frac{K_1(p)}{D_1(p)}$$

is greater than the degree of the other polynomials in (6.96).

Let condition (6.94) be satisfied exactly. The degenerate equation is

$$-K_1(p)K_2(p)D_3(p)F_{n2}(p)+K_1(p)K_2(p)K_3(p)F_{m2}(p)=0. \tag{6.97}$$

It is clear from (6.97) that if the transfer function Z_2 corresponds to positive local feedback (i.e., the signs are all as in equation (6.97)), the system is unstable. This conclusion is obtained from the following considerations. The first term in (6.97) is of higher degree than the second term. By results of Chapter Three the coefficients of the leading terms in the complete and the degenerate equation should have the same sign. Hence follows our first conclusion that the local feedback with the transfer function Z_2 is negative. The degenerate characteristic equation thus takes the form

$$K_1(p)K_2(p)D_3(p)F_{n2}(p)+K_1(p)K_2(p)K_3(p)F_{m2}(p)=0. \tag{6.98}$$

The conclusion concerning the feedback ratio Z_2 has been previously reported in /20/ and it is by no means new. We give it here only for the sake of completeness and consistency. Let us further assume that the parameters of the degenerate equation are so chosen that they satisfy the stability conditions. It is readily understood, however, that in the

case of perfect invariance the resulting system is not coarse in Andronov's sense and it is thus inadequate.

Indeed, suppose that the system departs from condition (6.94) by an infinitesimally small quantity. This gives rise to a small parameter which obviously may be either positive or negative. A negative parameter generates at least one right-half-plane root. In the case of perfect invariance we thus end up with a noncoarse, i.e., unrealizable, system.

It now remains to consider invariance in the case

$$\varepsilon > \frac{F_{n3}(p)}{F_{m3}(p)} - \frac{D_1(p)}{K_1(p)} = m > 0. \tag{6.99}$$

The characteristic equation is

$$mF_{N2}(p) + F_{N1}(p) = 0, \tag{6.100}$$

where

$$F_{N2}(p) = D_1(p)\,D_2(p)\,D_3(p)\,F_{m2}(p),$$
$$F_{N1}(p) = K_1(p)\,K_2(p)\,D_3(p)\,F_{n2}(p) + K_1(p)\,K_2(p)\,K_3(p)\,F_{m2}(p).$$

This system is realizable if it is structurally stable at infinite gain. In other words, we have proved that perfect invariance is unattainable in this way, while ε-invariance can be achieved only if the structure of the system has infinite-gain stability and the necessary and sufficient conditions of Chapter Three are satisfied.

We will not go into the corresponding results for multivariable control systems, since this would involve repetition of our previous discussion (see Chapter Three) of systems with infinite-gain stability. In conclusion of this section we will consider the properties of multivariable control systems based on the Watt—Polzunov principle, with the controlled variables interacting through the plant. It will be clear from what follows, however, that the nature of coupling is in general dependent not only on the properties of the plant but also on the structure of the control system itself.

To establish the dependence of coupling on structure, we divide the control systems into a number of structural groups according to the following signs:

(a) systems made up of one-loop single-variable subsystems;

(b) systems made up of one-loop single-variable subsystems with derivatives of from $(n-2)$-th to the first order inclusive delivered to the input (the system can be made stable when the subsystem gains increase indefinitely);

(c) systems made up of multiloop single-variable subsystems.

Let us consider each of these groups separately.

(a) Figure 6.14 is a block diagram of the control loop for the i-th variable. This structure is described by the following operator equation:

$$[D_i(p)R_i(p) + K_iK_{ci}]Y_i + K_iR_i(p)\sum_{i\neq k}\alpha_{ik}Y_k = K_iK_{ci}Y_{refi} + K_iR_i(p)f_i, \tag{6.101}$$

where $D_i(p)$ and K_i are the self-operator and the gain of the plant in respect to the i-th variable; $R_i(p)$ and K_{ci} ditto for the controller; α_{ik} the coefficient of coupling between the i-th and the k-th variables, determined by plant properties; Y_{iref} the reference value of the i-th controlled variable; f_i the load in the i-th loop.

We see from (6.101) that the coupling between the individual controlled variables depends not only on the properties of the plant (the coefficients α_{ik} and K_i), but also on the fundamental properties of the controller. For the sake of convenience, we divide (6.101) through by K_{ci}:

$$\left[\frac{1}{K_{ci}} D_i(p) R_i(p) + K_i\right] Y_i + \frac{K_i}{K_{ci}} R_i(p) \sum_{i \neq k} \alpha_{ik} Y_k = K_i Y_{\text{ref}\,i} + \frac{K_i}{K_{ci}} R_i(p) f_i. \quad (6.102)$$

The degree of coupling increases as K_i is increased and decreases as the controller gain K_{ci} increases; moreover, the dynamics of coupling depends on $R_i(p)$. We also see from (6.102) that the interrelationship between the controlled variables can be made arbitrarily small by appropriately increasing the controller gain K_{ci}, provided, of course, that the system remains stable in the process. If in the class of structures being considered $D_i(p)R_i(p)$ is of higher than second degree, the critical gain $K_{ci\,cr}$ should have a finite value, which determines the lower bound of coupling.

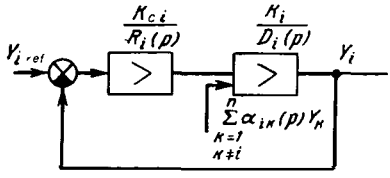

FIGURE 6.14. Disturbance rejection conditions for $K_c \to \infty$.

(b) To permit increasing the gains K_{ci} indefinitely without loss of the stability, derivatives of all orders from $(n-2)$-th to first inclusive (where n is the degree of the operator $D_i(p)R_i(p)$) are delivered to the system input. The equation for the i-th controlled variable then has the form

$$D_i(p) R_i(p) + K_{ci} K_i (a_0 p^{n-2} + a_1 p^{n-3} + \ldots + 1) Y_i + \\ + K_i R_i(p) \sum_{i \neq k} \alpha_{ik} Y_k = K_i K_{ci} Y_{\text{ref}\,i} + K_i R_i(p) f_i. \quad (6.103)$$

Dividing (6.103) through by K_{ci}, we find

$$\frac{1}{K_{ci}} D_i(p) R_i(p) + K_i (a_0 p^{n-2} + a_1 p^{n-3} + \ldots + 1) Y_i + \\ + \frac{K_i}{K_{ci}} R_i(p) \sum_{i \neq k} \alpha_{ik} Y_k = K_i Y_{\text{ref}\,i} - \frac{K_i}{K_{ci}} R_i(p) f_i.$$

In terms of coupling this case is not different from that under (a). It is significant, however, that the gain K_{ci} may increase indefinitely without incurring the danger of instability. The degree of coupling can therefore be made as small as desired.

In the two structures above decoupling is attained simultaneously with rejection of the external disturbances f_i.

(c) Finally we consider the case of multi-loop subsystems. Figure 6.15 is a block diagram of the multi-loop configuration for the i-th controlled variable. The corresponding equation is

$$\{[Q_i(p) F_{mi}(p) + K_{i\,\text{st}} F_{ni}(p)] N_i(p) D_i(p) + K_{i\,\text{tot}} F_{mi}(p)\} Y_i(p) + \\ + K_i \sum \alpha_{ik} Y_k(p) [Q_i(p) F_{mi}(p) + K_{i\,\text{st}} F_{ni}(p)] N_i(p) = \\ = K_{i\,\text{tot}} F_{mi}(p) Y_{i\,\text{ref}}(p) + \\ + K_i [Q_i(p) F_{mi}(p) + K_{i\,\text{st}} F_{mi}(p)] N_i(p) f_i(p), \quad (6.104)$$

where $K_{i\,tot} = K_i K_{Ni} K_{i\,st}$ and the following nomenclature is adopted for the i-th subsystem: $D_i(p)$ is the plant self-operator, $Q_i(p)$ the self-operator of the stabilized section, $N_i(p)$ the self-operator of the unstabilized section; $F_{ni}(p)$ and $F_{mi}(p)$ are respectively the numerator and the denominator operators of the stabilizer ratio; $K_{i\,st}$ is the gain of the stabilized section, K_i the plant gain, K_{Ni} the gain of the unstabilized section.

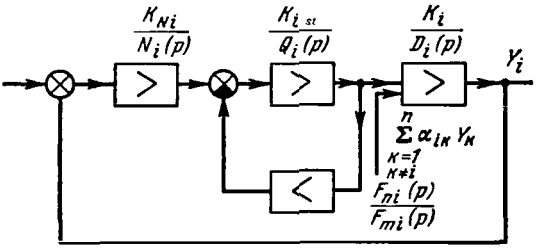

FIGURE 6.15. Illustrating the disturbance rejection conditions in a more complex case.

To simplify further analysis, we write (6.104) in the form

$$\left\{\left[\frac{1}{K_{st}} Q_i(p) F_{mi}(p) + F_{ni}(p)\right] N_i(p) D_i(p) + K_{Ni} K_i F_{mi}(p)\right\} Y_i +$$
$$+ \frac{K_i}{K_{i\,st}} Q_i(p) N_i(p) F_{mi}(p) \sum \alpha_{ik} Y_k +$$
$$+ K_i F_{ni}(p) N_i(p) \sum_{i \neq k} \alpha_{ik} Y_k = K_i K_{Ni} F_{mi}(p) Y_{i\,ref} +$$
$$+ \frac{K_i}{K_{i\,st}} [Q_i(p) F_{mi}(p) + K_{i\,st} F_{ni}(p)] N_i(p) f_i. \qquad (6.105)$$

We see from (6.105) that the dependence of coupling on structure in this general case is determined by the two components of equation (6.105) which contain the sums $\sum \alpha_{ik} Y_k$. In the first component the coupling coefficient is directly proportional to the plant gain for the i-th controlled variable, dependent on the controller self-operator and the denominator operator of the stabilizer ratio, and inversely proportional to the gain of the unstabilized section.

In the second component the coupling coefficient is proportional to the plant gain and dependent on the self-operator of the unstabilized part of the controller and on the numerator operator of the stabilizer ratio.

In systems with infinite-gain stability, the first component can be reduced to a minimum by increasing the gain parameter $K_{i\,st}$. The second component has noticeable influence on the dynamics of the process, since under steady-state conditions $F_{ni}(p)_{p=0} = 0$.

A similar pattern is obtained when stabilization is achieved by ideal derivatives embracing only a part of the control circuit. The corresponding equation can be derived from (6.105) by putting $F_{mi}(p) = 1$.

We have thus established that, although the individual controllers are not interrelated, the coupling of the controlled variables is highly sensitive to the structure of the single-variable subsystems. External disturbances can be rejected only by increasing the overall gain.

§ 6.8. NONINTERACTION AND INVARIANCE IN THE GENERAL CASE OF A MULTIVARIABLE COMBINED-CONTROL SYSTEM

In this section we proceed with a discussion of multivariable control systems with the variables interconnected through the plant and through the load. In previous sections we have established that invariance in systems of this kind does not necessarily imply noninteraction, and vice versa: noninteraction does not automatically ensure invariance. We will now consider some methods that ensure noninteraction and invariance simultaneously.

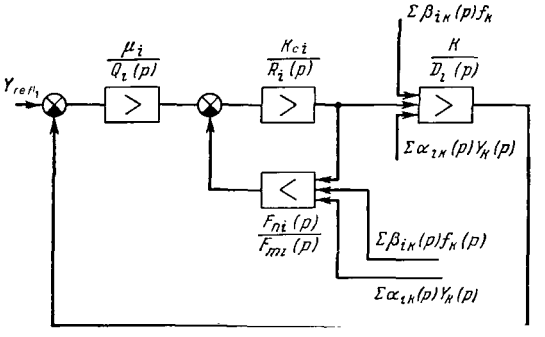

FIGURE 6.16. Illustrating simultaneous derivation of noninteraction and invariance.

We have seen that invariance to an accuracy of ε is achieved in structures with infinite-gain stability by applying an additional disturbance signal to the general stabilizer input. We have also seen that noninteraction for structures of this kind is attained by additionally delivering to the general stabilizer input the sum of all extraneous controlled variables, each multiplied by the respective coefficient $\alpha_{ik}(p)$.

Let us now consider how to simultaneously achieve noninteraction and invariance. We will establish the additional restrictions to be imposed on the system structure and parameters in this case.

Figure 6.16 is a block diagram of the i-th subsystem in a multivariable combined-control system. We see from Figure 6.16 that the sum of the extraneous controlled variables is delivered to the stabilizer input. The element with the transfer function $\frac{K_{ci}}{R_i(p)}$ additionally receives the sum of all the external disturbances (through a transducer). This structure is a logical outgrowth of the configurations considered in the previous chapter. As before, we assume that the system is structurally stable at infinite gain.

Without repeating the elementary manipulations, we write the matrix equation for the case on hand:

$$Y_j = A^{-1}(p)[BY_{\text{ref}} + LF]. \tag{6.106}$$

Inserting for the matrices their explicit expressions and multiplying we find

$$Y(p)=\frac{1}{\Delta}\begin{Vmatrix}\sum_{k=1}^{n}A_{1k}K_{c\,k}l_{kk}Y_{ref\,k}\\ \sum_{k=1}^{n}A_{kj}K_{c\,k}l_{kk}Y_{ref\,k}\\ \sum_{k=1}^{n}A_{kn}K_{c\,k}l_{kk}Y_{ref\,k}\end{Vmatrix}+\frac{1}{\Delta}\begin{Vmatrix}\sum_{k=1}^{n}A_{k1}[K_{c\,k}(c_{kk}-\rho_{kk})+d_{kk}]\sum_{l=1}^{n}\beta_{1l}f_l\\ \sum_{k=1}^{n}A_{kj}[K_{c\,k}(c_{kk}-\rho_{kk})+d_{kk}]\sum_{l=1}^{n}\beta_{jl}f_l\\ \sum_{k=1}^{n}A_{kn}[K_{c\,k}(c_{kk}-\rho_{kk})+d_{kk}]\sum_{l=1}^{n}\beta_{jl}f_l\end{Vmatrix}. \quad (6.107)$$

For the j-th controlled variable we have

$$Y_j(p)=\frac{1}{\Delta}\left\{\sum_{k=1}^{n}A_{kj}K_{c\,k}l_{kk}Y_{ref\,k}+\sum_{k=1}^{n}A_{kj}[K_{c\,k}(c_{kk}-\rho_{kk})+d_{kk}]\sum_{l=1}^{n}\beta_{jl}f_l\right\}. \quad (6.108)$$

Here

$$\Delta=\begin{vmatrix}a_{11}+K_{c\,1}b_{11} & d_{11}a_{12} & \dots & d_{11}a_{1j} & \dots & d_{11}a_{1n}\\ d_{22}a_{21} & a_{22}+K_{c\,2}b_{22} & \dots & d_{22}a_{2j} & \dots & d_{22}a_{2n}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ d_{nn}a_{n1} & \dots & \dots & d_{nn}a_{nj} & \dots & a_{nn}+K_{c\,n}b_{nn}\end{vmatrix},$$

A_{kj} is obtained from the transpose A_t,

$$\left.\begin{aligned}a_{ll}&=Q_lR_l'D_lF_{ml}\Theta_{ml},\\ b_{ll}&=M_lK_lF_{ml}\Theta_{ml}+F_{nl}Q_lD_l\Theta_{ml},\\ c_{ll}&=K_lF_{nl}Q_l\Theta_{ml},\\ d_{ll}&=K_lQ_lR_l'F_{ml}\Theta_{ml},\\ l_{ll}&=K_lM_{ll}F_{ml}\Theta_{ml},\\ \rho_{ll}&=K_l\Theta_{nl}Q_lF_{ml}.\end{aligned}\right\} \quad (6.109)$$

Let us first consider invariance of the controlled variable Y_i under the disturbances. Invariance is ensured if

$$K_{c\,k}(c_{kk}-\rho_{kk})+d_{kk}=0. \quad (6.110)$$

Inserting for c_{kk}, ρ_{kk}, and d_{kk} their expressions from (6.109), we write

$$K_{c\,k}(K_kF_{nk}Q_k\Theta_{mk}-K_k\Theta_{nk}Q_kF_{mk})+K_kQ_kR_k'F_{mk}\Theta_{mk}=0,$$

or

$$K_kQ_k(F_{nk}\Theta_{mk}-F_{mk}\Theta_{nk})+\frac{K_kQ_kR_kF_{mk}\Theta_{mk}}{K_c}=0. \quad (6.111)$$

Since by assumption the structure is stable for $K_c\to\infty$, the conditions of ε-invariance for sufficiently high K_c are still written in the same form as in Chapter Five:

$$\frac{F_{nk}}{F_{mk}}=\frac{\Theta_{nk}}{\Theta_{mk}}. \quad (6.112)$$

Now consider the noninteraction conditions. Noninteraction of the j-th controlled variable is ensured, i.e., the controlled variable Y_j is made independent of all Y_k, $k=1,\dots,n$, $k\neq j$, in both the steady-state and the

transient modes of operation if the determinant Δ in (6.108) is independent of the coupling coefficients α_{ih}; an additional requirement is that the right-hand side of the equation should contain terms with $Y_{\text{ref}\,j}$ only, while terms with $Y_{\text{ref}\,k}$, $k=1, 2, \ldots, n$, $k \neq j$, all vanish.

The product $\prod_{l=1}^{n} K_{c\,l}$ can be taken outside the determinant Δ. Putting $\frac{1}{K_{c\,l}} = m_l$, we write

$$\Delta = \prod_{l=1}^{n} K_{c\,l} \begin{vmatrix} m_1 a_{11} + b_{11} & m_1 d_{11}\alpha_{12} & m_1 d_{11}\alpha_{1j} & m_1 d_{11}\alpha_{1n} \\ m_2 d_{22}\alpha_{21} & m_2 a_{22} + b_{22} & m_2 d_{22}\alpha_{1j} & m_2 d_{22}\alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ m_j d_{jj}\alpha_{j1} & \cdots & m_j a_{jj} + b_{jj} & m_j a_{jj}\alpha_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ m_n d_{nn}\alpha_{n1} & \cdots & m_n d_{nn}\alpha_{nj} & m_n a_{nn} + b_{nn} \end{vmatrix}. \tag{6.113}$$

If the necessary and sufficient conditions of stability /39/ are satisfied, the determinant (6.113) for sufficiently large $K_{c\,i}$ ($m_i \to 0$) degenerates to

$$\Delta = \begin{vmatrix} b_{11} & 0 & \ldots & 0 & \ldots & 0 \\ 0 & b_{22} & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & & b_{jj} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & & \ldots & & b_{nn} \end{vmatrix}. \tag{6.114}$$

If for the time being we ignore the factor $\prod_{l=1}^{n} K_{c\,l}$, the left-hand side of (6.108) separates into n factors and the determinant is independent of the coupling coefficients α_{ik}.

Consider the right-hand side of (6.108). If the invariance conditions are satisfied, it has the form

$$\sum_{k=1}^{n} A_{kj} K_{c\,k} l_{kk} Y_{\text{ref}\,k}. \tag{6.115}$$

Let us derive explicit expressions for A_{kj} in the case $K_{c\,k} \to \infty$. The transpose has the form

$$A_{kj} = \begin{Vmatrix} a_{11} + K_{c\,1} b_{11} & d_{22}\alpha_{21} & d_{j1}\alpha_{j1} & d_{nn}\alpha_{n1} \\ d_{11}\alpha_{12} & a_{22} + K_{c\,2} b_{22} & d_{j2}\alpha_{j2} & d_{nn}\alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ d_{11}\alpha_{1j} & d_{22}\alpha_{2j} & a_{jj} + K_{c\,j} b_{jj} & d_{nn}\alpha_{nj} \\ \cdots & \cdots & \cdots & \cdots \\ d_{11}\alpha_{1n} & d_{22}\alpha_{2n} & d_{nn}\alpha_{jn} & a_{nn} + K_{c\,n} b_{nn} \end{Vmatrix}. \tag{6.116}$$

For the j-th controlled variable Y_j the elements of the j-th row in the transpose are replaced by their cofactors, which are the A_{kj}.

Now the cofactors have the following obvious properties:

(a) In the nonsingular case the rank of the cofactor is one less than the rank of the system determinant.

(b) In each cofactor A_{kj} ($k \neq j$) there is at least one row which contains no elements with $K_{c\,l}$, and it is only in A_{jj} that each row contains an element with $K_{c\,l}$.

Making use of these properties of cofactors and employing equation (6.113), we write for the j-th controlled variable

$$Y_j = \frac{\prod_{l=2}^{n} K_{c\,l} l_{11} K_{c\,1} \,|\, m \,|\, Y_{\text{ref}\,1}}{D_1} + \ldots + \frac{\prod_{\substack{l=1 \\ l \neq j}}^{n} K_{c\,l} l_{jj} K_{c\,j} \begin{vmatrix} m_1 a_{11} + b_{11} & \ldots & m_1 d_{nn}\alpha_{n11} \\ m_n d_{nn}\alpha_{1n} & \ldots & m_n a_{nn} + b_{nn} \end{vmatrix} Y_{\text{ref}\,j}}{D_j}, \tag{6.117}$$

where

$$D_1 = \prod_{i=1}^{n} K_{ci} \begin{vmatrix} m_1 a_{11} + b_{11} & m_{11} d_{11} a_{12} & \dots & m_1 d_{11} a_{1n} \\ \dots & \dots & \dots & \dots \\ m_j d_{jj} a_{j1} & \dots & \dots & m_j d_{jj} a_{jn} \\ \dots & \dots & \dots & \dots \\ m_n d_{nn} a_{n1} & \dots & \dots & m_n a_{nn} + b_{nn} \end{vmatrix}.$$

Here $|m|$ is a determinant with all the elements of at least one row multiplied by m; dots in the numerator in the explicitly written determinant $|\tilde{m}|$ indicate that elements multiplied by $|m|$ are to follow. It is also clear from (6.117) that all the terms contain a common factor $\prod_{i=1}^{n} K_{ci}$. Dividing the numerator and the denominator by $\prod_{i=1}^{n} K_{ci}$ and passing to the limit as $m \to 0$, we find

$$Y_j = \frac{l_{jj} \begin{vmatrix} b_{11} & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & b_{22} & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & b_{j-1,j-1} & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & b_{j+1,j+1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & b_{nn} \end{vmatrix}}{\begin{vmatrix} b_{11} & 0 & 0 & \dots & 0 \\ 0 & b_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & b_{nn} \end{vmatrix}} Y_{\text{ref } j} = \frac{l_{jj} Y_{\text{ref } j}}{b_{jj}}. \qquad (6.118)$$

We have thus obtained noninteraction of the j-th controlled variable to an accuracy of ε. Hence readily follows the conclusion that if the structure in Figure 6.16 is stable at infinite gain and the necessary and sufficient conditions of stability are satisfied, this structure can be made to ensure simultaneous noninteraction and invariance at sufficiently high gain.

Chapter Seven

SYNTHESIS OF FIXED-STRUCTURE SYSTEMS WITH PROPERTIES EQUIVALENT TO ADAPTIVE SYSTEMS

§ 7.1. INTRODUCTION

Adaptive (self-adjusting) systems are used when the control system is expected to alter its properties in accordance with the properties of the controlled object. This situation may arise in the following typical cases:

(a) The plant parameters change in the process of control. The variation of plant parameters may be due to external disturbances which cannot be programmed beforehand or to some change in the operating conditions of the plant. The structure and the parameters of the control system, though quite adequate for the initial state of the plant, may prove to be inadequate under the new conditions.

(b) There is an extensive class of controlled objects whose output has an extremum in relation to a certain quality criterion. The problem to be solved in the design of control systems for these objects is how to find the extremum and how to hold it by on-line variation of system parameters, so that, irrespective of external disturbances, the plant always remains on the optimum frequency response (optimum operating conditions). The control system is provided with an appropriate searching circuit, which is generally a fairly complex device. Searching control systems are also used when no information is available on the plant properties or when only partial information is at hand.

In any case the system should have the property of adaptation or self-adjustment. Sometimes simple adjustment of the numerical values of some system parameters is insufficient, and adaptation is attained by actually changing the structure of the system.

Systems with a self-improving program are somewhat different from adaptive systems. Here, the ordinary feedback logic is insufficient for effective control. The characteristic features of this case are best illustrated by the following example.

In Chapter One we considered the continuous hot-rolling mill as a typical example of a multivariable controlled object. The principal aim of the control system was to ensure constant thickness of the rolled sheet. The sheet thickness is a function of many variables. In the general case, the deviation from a given gage can be expressed as

$$\alpha = F(h_i, T, n_i, n_{i+1}, t, \delta, \tau_i), \tag{7.1}$$

where h_i are the roll coordinates, T is the rolling temperature, n_i, n_{i+1} are the rolling speeds in the i-th and the $(i+1)$-th stands, τ_i is the time lag, δ is a random variable, depending on the condition of the mill, uniformity of the metal, and other random factors.

The problem is to choose the variables entering the function F so as to minimize the gage deviation α (ideally it should be zero) and to maintain it between permissible limits.

It is clear that (7.1) is a functional and we have here a variational problem. It is also fairly obvious that we are dealing with a problem in multivariable control. Indeed, the sheet thickness can be altered by changing the roll gap or by adjusting the strip tension. These two methods of gage control, however, are not independent. We know from the theory of rolling and from numerous experiments that the variation of roll gap effectively alters the strip thickness only if the interstand tension remains constant. If now the roll gap is adjusted without controlling the rolling tension, the thickness will change insignificantly and there is moreover the danger of looping on the reduction end of the stand and stretching (or even rupture) of the strip on the other end. This development must be avoided at all costs, so that the roll gaps and the rolling speeds are controlled simultaneously.

The roll gap is controlled through the pressing screws, which are positioned by a special regulator in each stand; the roll speeds are adjusted by controlling the main drives of each stand. The different control systems are interconnected through the strip. Hot-rolling gage control has another characteristic feature which requires a special approach to the design of the system. The strip thickness can be directly measured only after the last stand; transportation lag makes it impossible to act on the strip section that is being measured at the given time. This is the main reason why an ordinary deviation-control system will not do in hot-rolling mills.* However, the distribution of thickness variability along the strip is nearly the same for the i-th and the $(i+1)$-th strip, while for nonconsecutive strips this distribution may be essentially different. This hypothesis is borne out by a wealth of statistical data and constitutes the basis of what is known as systems with self-improving program.

In these systems, the rolling program for the i-th strip is developed from the measurement data for all the previous $i-1$ strips. Rolling-mill control processes are of independent interest; here they are discussed only as another example of adaptive systems.

We see that adaptation or self-adjustment is required when the plant properties change due to external or internal disturbances and when incomplete information is available on the controlled object. An indispensable component of such adaptive systems (with the exception of the last one considered in this section) is a searching circuit, and adaptation is achieved by an adjustment of the system parameters or even modification of the system structure to meet a certain quality criterion. An interesting question to be considered in this context is the synthesis of self-adjusting systems which adapt without requiring a change in structure. In the following we shall see that such fixed self-adjusting structures can be designed for a sufficiently large class of controlled objects. The subject of this chapter is thus fixed-structure systems which have the same properties as adaptive systems.

* Deviation control can be instituted by regulating some indirect parameter, e.g., the roll pressure.

§ 7.2. STRUCTURAL NOISE REJECTION IN A CERTAIN CLASS OF DYNAMIC SYSTEMS

As we have previously noted, one of the reasons for the variation in plant properties is the presence of external disturbances. A highly topical problem of modern automatic control theory and practice is the choice of structures and parameters which are as noiseproof as possible.

Considerable attention is devoted in the literature to the problem of noise suppression (see, e.g., /66/). The case considered in /66/ is that of a noisy input, when the aim is to isolate the effective signal against the background of noise.

In automatic control systems and in a number of servosystems the effective signal or the reference pulse are without noise. Noise is injected in several points along the control channel. This noise is contributed by various loads and disturbances, which may be of a random nature. In this section we deal with the case of random noise and show how to choose the structure and the parameters of a control system so as to minimize the interference. In the beginning we consider a single-variable system, and then generalize the results to multivariable control systems.

1. Single-variable control system

Figure 7.1 is a block diagram of an automatic control system. The reference signal Y_{ref} is delivered to the input and the system is expected to reproduce this signal faithfully. It is assumed that Y_{ref} is noisefree.

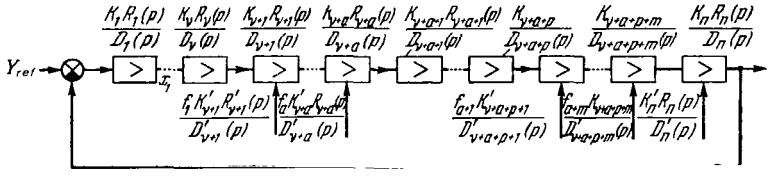

FIGURE 7.1. The general case of an N-element system.

The system consists of N elements with transfer functions $\frac{K_i R_i(p)}{D_i(p)}$. Of these N elements, $a+m$ elements in different parts of the closed loop are noisy. For the sake of simplicity, let a noisy elements be concentrated in one part of the loop, and the other m elements in some other part, so that β noisefree elements separate between the two groups a and m. This particular setup is adopted in order to simplify the mathematical manipulations. The conclusions, however, are quite general and can be applied for an arbitrary distribution of noisy elements in the control loop. There is only one condition imposed on the position of the elements. If the system input or origin is the point where the reference signal is delivered (we have already remarked that the reference signal is noisefree), then the first ν elements after the input (where ν is any nonzero integer) are without

noise. Since noise is not necessarily injected at the input of each element, the transfer function between the point of noise injection and the output of the element is regarded as being different from the transfer function of the element (we denote it by $\frac{K_i'R_i'(p)}{D_i'(p)}$).

There is only one restriction imposed on noise: the noise and all its time derivatives have a finite absolute value, i.e.,

$$|f_i^{(k)}| < M \qquad (i=1, 2, \ldots, n;\ k=0, 1, \ldots, m). \tag{7.2}$$

Otherwise, the noise may be represented by any, in particular random, function of time.

We now prove some properties of this class of structures, which are jointly referred to as "structural noise rejection".

The accuracy of reproduction of the input reference signal Y_{ref} increases as the gain of the noisefree elements increases. Noise is suppressed by the gain of the elements immediately preceding the noisy components of the dynamic chain.

To prove this proposition, we have to find the transfer function of the system shown in Figure 7.1.

For the noisefree elements we may write

$$x_{i+1} = \frac{K_iR_i(p)}{D_i(p)}x_i. \tag{7.3}$$

For the noisy elements,

$$x_{j+1} = \frac{K_jR_j(p)}{D_j(p)}x_j + \frac{K_j'R_j'(p)}{D_j'(p)}f_j. \tag{7.4}$$

After some calculations, we obtain the following expression for the output of the loop shown in Figure 7.1:

$$\begin{aligned}\left[\prod_{i=1}^{\nu+\alpha+\beta+m}\frac{K_iR_i(p)}{D_i(p)}\frac{K_nR_n(p)}{D_n(p)}+1\right]Y_{out} &= \prod_{i=1}^{\gamma+\alpha+\beta+m}\frac{K_iR_i(p)}{D_i(p)}\frac{K_nR_n(p)}{D_n(p)}Y_{ref}+\\ &+\frac{K_nR_n(p)}{D_n(p)}\prod_{i=\alpha+1}^{\beta+m}\frac{K_iR_i(p)}{D_i(p)}\sum_{j=2}^{\alpha+j}\prod_{\rho=1}^{\alpha}\frac{K_{\rho+\nu}R_{\rho+\nu}(p)}{D_{\nu+\rho}(p)}\frac{K'_{\nu+j-1}R'_{\nu+j-1}(p)}{D'_{\nu+j-1}}f_{j-1}+\\ &+\frac{K_nR_n(p)}{D_n(p)}\sum_{j=2}^{m+1}\prod_{\rho=1}^{m}\frac{V_{\nu+\alpha+\beta+i}(p)}{D_{\nu+\alpha+\beta+j}(p)}\frac{K'_{\nu+\alpha+\beta+j}R'_{\nu+\alpha+\beta+j-1}(p)}{D'_{\nu+\alpha+\beta+j-1}(p)}f_{\alpha+j-1}+\frac{K_n'R_n'(p)}{D_n'(p)}f_n.\end{aligned} \tag{7.5}$$

Let the gain of the noisefree elements be sufficiently high. Then dividing (7.5) through by $\prod_{i=1}^{\nu}K_i\prod_{j=\nu+\alpha+1}^{\beta}K_j$ and putting $\frac{1}{K_i}=\frac{1}{K_j}=m$, we obtain after simple manipulations

$$\begin{aligned}&\left[m^{\nu+\beta}\prod_{i=1}^{\nu+\alpha+\beta+m}D_i(p)\prod_{j=\nu+1}^{\alpha}D_j'(p)\prod_{\rho=\nu+\alpha+\beta+1}^{m}D_\rho'(p)D_n(p)+\right.\\ &+K_nR_n(p)D_n'(p)\prod_{i=\nu+1}^{\alpha}K_iR_i(p)\prod_{j=\nu+1}^{\alpha}D_i'(p)\prod_{j=\nu+\alpha+\beta+1}^{n}K_jR_j(p)\times\\ &\left.\times\prod_{\rho=\nu+\alpha+\beta+1}^{m}D_\rho'(p)\right]Y_{out}=K_nR_n(p)D_n'(p)\prod_{i=\nu+1}^{\alpha}K_iR_i(p)\times\end{aligned}$$

$$\times \prod_{j=\nu+\alpha+\beta+1}^{\alpha} K_j R_j(p) \prod_{j=\nu+1}^{\alpha} D_j'(p) \prod_{\rho=\nu+\alpha+\beta+1}^{m} D_\rho'(p) Y_{\text{ref}} +$$

$$+ m^\nu K_n R_n(p) \prod_{l=\alpha+1}^{\beta+m} R_l(p) \sum_{\rho=1}^{\alpha} \prod K_{\nu+\rho} R_{\nu+\rho}(p) K'_{\nu+\rho-1} R'_{\nu+j-1}(p) \times$$

$$\times \prod_{i=1}^{\nu+\alpha} D_i(p) \left\{ D_{\nu+1}(p) + D_{\nu+2}(p) D_{\nu+1}(p) + \ \dots \ + \prod_{\nu=1}^{\alpha+\beta} D_\nu(p) \right\} \times$$

$$\times \prod_{i=\nu+1}^{\alpha} D_i'(p) \prod_{j=\nu+\alpha+\beta+j}^{m} D_j'(p) f_{j-1} + m^{\nu+\beta} K_n R_n(p) \prod_{i=1}^{\nu+\alpha+\beta} D_i(p) \times$$

$$\times \left[D_{\nu+\alpha+\beta}(p) + D_{\nu+\alpha+\beta+1}(p) + \ \dots \ + \prod_{j=1}^{m} D_{\nu+\alpha+\beta+j}(p) \right] \times$$

$$\times \sum_{j=2}^{m+1} \prod_{\rho=1}^{m} K_{\nu+\alpha+\beta+j} R_{\nu+\alpha+\beta+j}(p) \prod_{i=\nu+1}^{\alpha} D_i'(p) \prod_{j=\nu+\alpha+\beta+j} D_j'(p) \times$$

$$\times K'_{\nu+\alpha+\beta+j-1} R'_{\nu+\alpha+\beta+j-1}(p) f_{\alpha+j-1} + m^{\nu+\beta} \prod_{i=1}^{\nu+\alpha+\beta+m} D_j(p) \times$$

$$\times \prod_{i=\nu+1}^{\alpha} D_i(p) \prod_{j=\nu+\alpha+\beta+1}^{m} D_i(p) K_n' R_n'(p) f_n. \qquad (7.6)$$

From (7.6), clearly,

$$\lim_{m \to 0} Y_{\text{out}} = Y_{\text{ref}}, \qquad (7.7)$$

if this limit exists, i.e., if the system is realizable (stable) /39/. We have thus proved the following:

(1) for the class of structures being considered noise can be suppressed by increasing the gain of the noisefree elements;

(2) noise suppression improves for noisy elements far from the input.

The general equation of the output variable for structures of this class can be written as

$$[m^{\nu+\beta+\gamma+} \cdots F_{N0}(p) + R_0(p)] Y_{\text{out}} = R_0(p) Y_{\text{ref}} + m^\nu \sum_i F_{\nu+1}(p) f_l +$$

$$+ m^{\nu+\beta} \sum_j F_{\nu+\beta+j}(p) f_l + \ \dots \ + m^{\nu+\beta+\gamma+} \cdots F_n(p) f_n; \qquad (7.8)$$

for $m \to 0$

$$Y_{\text{out}} = Y_{\text{ref}}.$$

Such a system is realizable if and only if it remains stable for $m \to 0$, i.e., for $K_i \to \infty$. In other words, if the system retains its stability despite an indefinite increase in the gain of the noisefree elements, the reference signal can be reproduced with infinitely high accuracy. The structural aspect of this proposition is that noise is suppressed by the gain of the elements which are situated between the input of the control system and the noisy element.

2. Multivariable control systems

Figure 7.2 is a block diagram of the i-th control loop. The plant and another element of the circuit are noisy. That the loop includes only four and not N elements obviously does not affect the generality.

The relevant equations in Laplace transforms are

$$X_1 = \frac{K_{1i}}{D_{1i}(p)} Y_{\text{ref}\,i} - \frac{K_{1i}}{D_{1i}(p)} Y_i, \tag{7.9}$$

$$X_2 = \frac{K_{2i}}{D_{2i}(p)} X_1 + \frac{K'_{2i}}{D'_2(p)} f_1, \tag{7.10}$$

$$X_3 = \frac{K_{3i}}{D_{3i}(p)} X_2, \tag{7.11}$$

$$Y_i = \frac{K_{4i}}{D_{4i}(p)} X_3 + \frac{K_{4i}}{D_{4i}(p)} \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k + \frac{K'_4}{D'_4(p)} f_2. \tag{7.12}$$

Solving equations (7.9) – (7.12) for Y_i, we find

$$\begin{aligned} &\left[\prod_{k=1}^{4} D_{ik}(p)\, D'_{2i}(p)\, D'_{4i}(p) + K_{1i}K_{2i}K_{3i}K_{4i}D'_{2i}(p)\, D'_{4i}(p)\right] Y_i + \\ &\quad + D_{1i}(p)\, D_{2i}(p)\, D_{3i}(p)\, D'_{2i}(p)\, D'_{4i}(p)\, K_{4i} \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k = \\ &\quad = K_{1i}K_{2i}K_{3i}K_{4i}D'_{2i}(p)\, D'_{4i}(p)\, Y_{\text{ref}\,i} + \\ &\quad + D_{1i}(p)\, D'_{4i}(p)\, K'_{2i}K_{3i}K_{4i}f_{1i} + D_{1i}(p)\, D_{2i}(p)\, D_{3i}(p)\, D'_{2i}(p)\, K'_{4i}f_{2i}. \end{aligned} \tag{7.13}$$

Dividing both sides of (7.13) by K_1K_3 and putting

$$\frac{1}{K_{1i}} = \frac{1}{K_{3i}} = m,$$

we obtain

$$\begin{aligned} &\left[m^2 \prod_{k=1}^{4} D_{ik}(p)\, D'_{2i}(p)\, D'_{4i}(p) + K_{2i}K_{4i}D'_{2i}(p)\, D'_{4i}(p)\right] Y_i + \\ &\quad + m^2 D_{1i}(p)\, D_{2i}(p)\, D_{3i}(p)\, D'_{2i}(p)\, D'_{4i}(p)\, K_4 \sum \alpha_{ik} Y_k = \\ &\quad = K_{2i}K_{4i}D'_{2i}(p)\, D'_{4i}(p)\, Y_{\text{ref}\,i} + mD_{1i}(p)\, D_{2i}(p)\, D'_{4i}(p)\, K_{2i}K_{4i}f_{1i} + \\ &\quad + m^2 D_{1i}(p)\, D_{2i}(p)\, D_{3i}(p)\, D_{4i}(p)\, D'_{2i}(p)\, K_{4i}f_{2i}, \end{aligned} \tag{7.14}$$

and in the limit of very high gain

$$\lim_{m \to 0} Y_i = Y_{i\,\text{ref}}. \tag{7.15}$$

We have obtained the same result and the same structural property. Thus, the disturbance f_{1i} is close to the input and only the gain of the first element is available for its suppression. The disturbance f_{2i} is far enough from the input to be suppressed by the gains K_{1i} and K_{3i}, etc.

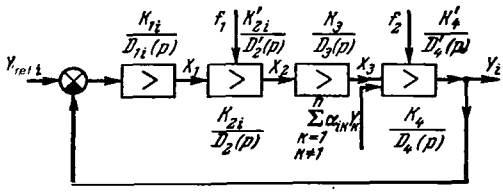

FIGURE 7.2. Illustrating structural noise rejection.

Structural noise rejection is thus a property that can be readily extended to multivariable control systems. Here again system realizability is obviously connected with considerations of infinite-gain stability.

§ 7.3. PHYSICAL REALIZABILITY OF SYSTEMS WITH STRUCTURAL NOISE REJECTION

First let us consider the realizability of noiseproof structures in application to single-variable systems (Figure 7.3). The noisy elements are the second and the fourth. The gain of the noisefree elements can be manipulated as required.

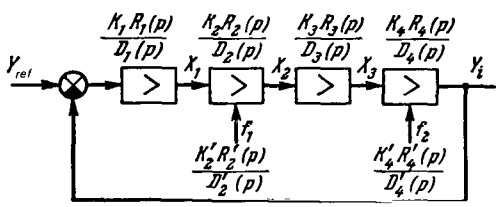

FIGURE 7.3. Noise rejection in a more general case.

A structure is said to be realizable if it remains stable at infinite gain and if the noise can be suppressed by increasing the gain. * It will be seen from the following that noise rejection is possible only if the high-gain element is unstabilized, so that the questions of stability and realizability should be considered separately in this case.

After appropriate calculations, we obtain for the output variable of the structure shown in Figure 7.3

$$Y_{out} = \frac{\prod_{i=1}^{4} R_i(p)\, D_2'(p)\, D_4'(p)\, K_1K_2K_3K_4 Y_{ref}}{\prod_{i=1}^{4} D_i(p)\, D_2'(p)\, D_4'(p) + D_2'(p)\, D_4'(p) \prod_{i=1}^{4} R_i(p)\, K_1K_2K_3K_4} + \\ + \frac{P_1}{\prod_{i=1}^{4} D_i(p)\, D_2'(p)\, D_4'(p) + D_2'(p)\, D_4'(p) \prod_{i=1}^{4} R_1(p)\, K_1K_2K_3K_4}, \qquad (7.16)$$

where

$$P_1 = \prod_{i=1}^{4} D_i(p)\, D_2'(p)\, K_4'R_4' f_2 + D_1(p)\, D_2(p)\, D_4'(p)\, K_2'(p)\, R_2'(p)\, K_3R_3(p)\, K_4R_4(p)\, f_1.$$

System stability depends on the position of the poles of the right-hand side of (7.16) or, equivalently, on the zeros of the characteristic equation

$$\prod_{i=1}^{4} D_i(p)\, D_2'(p)\, D_4'(p) + D_2'(p)\, D_4'(p) \prod_{i=1}^{4} R_i(p)\, K_1K_2K_3K_4 = 0. \qquad (7.17)$$

* A more rigorous discussion of realizability, taking account of system coarseness requirements, is given in a special section in what follows.

Dividing (7.17) through by K_1K_3 and putting

$$\frac{1}{K_1}=\frac{1}{K_3}=m,$$

we write

$$\left[m^2\prod_{i=1}^{4}D_i(p)+K_2K_4\prod_{i=1}^{4}R_i(p)\right]D_2'(p)\,D_4'(p)=0. \tag{7.18}$$

The system is realizable if and only if the roots of (7.18) are in the left-half plane for $m\to 0$. Now, we see from (1.18) that for this to hold true it is necessary and sufficient that the roots of each factor in (7.18) are in the left-half plane for $m\to 0$. Since the roots of $D_2'(p)\,D_4'(p)$ are independent of m, the roots that they generate depend on the self-operator of the noisy elements. We assume that the elements are intrinsically stable, and $D_2'(p)\,D_4'(p)=0$ therefore has left-half-plane roots. The stability of the system as a whole therefore depends on the roots of the second factor

$$m^2\prod_{i=1}^{4}D_i(p)+K_2K_4\prod_{i=1}^{4}R_i(p)=0 \tag{7.19}$$

for $m\to 0$.

The results of Chapter Three suggest the following procedure for the determination of the stability conditions.

Let N_2 be the degree of the polynomial $\prod_{i=1}^{4}D_i(p)$ and N_1 the degree of the polynomial $\prod_{i=1}^{4}R_i(p)$; the system is stable if

(1) $N_2-N_1\leqslant 2$;

(2) $K_2K_4\prod_{i=1}^{4}R_i(p)=0$ satisfies the stability conditions;

(3) certain relationships are observed between the coefficients of the polynomials $\prod_{i=1}^{4}D_i(p)$ and $K_2K_4\prod_{i=1}^{4}R_i(p)$ depending on which of the two following equalities is true:

$$N_2-N_1=1$$

or

$$N_2-N_1=2.$$

Let us consider the most difficult cases as far as realizability is concerned.

Suppose that the elements shown in Figure 7.3 are made up of aperiodic components. If out of the total of N components, ν are high-gain devices, equation (7.19) takes the form

$$m^{\nu}\prod_{i=1}^{N}(1+T_ip)+\prod_{i=1}^{n}K_i\prod_{j=n+\nu+1}^{N}K_j=0.$$

Putting $\prod_{l=1}^{n} K_l \prod_{j=n+\nu+1}^{N} K_j = K_{\text{deg}}$, we obtain

$$m^{\nu} \prod_{l=1}^{N} (1 + T_l p) + K_{\text{deg}} = 0. \tag{7.20}$$

Equation (7.20) clearly satisfies the stability conditions for $m \to 0$ only if $N \leqslant 2$. This is a trivial case of very limited interest.

Let us consider stabilization of the system for $N_2 > 2$.

System (7.20) is stabilized for $m \to 0$ by feeding into it derivatives at least from the $(N-2)$-th to the first order. We now modify the structure of the system by introducing additional $N-2$ amplifiers (the gains of these amplifiers can be made sufficiently high). Each of these amplifiers is enclosed in a negative feedback loop with a transfer function $\frac{M_l}{1+T_l p}$ (Figure 7.4). As regards the remaining part of the circuit, we assume that ν out of the N aperiodic components are noisefree, and that their gain can be varied between wide limits.

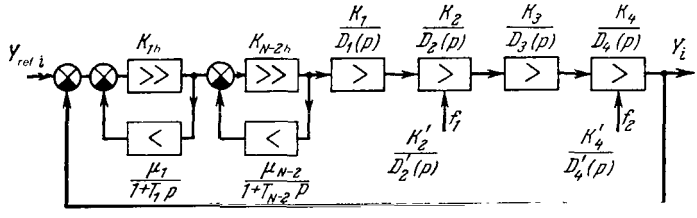

FIGURE 7.4. Structure ensuring stability and noise rejection.

For the sake of simplicity the noisy elements are collected in two groups, which are located as shown in Figure 7.4.

For the first $N-2$ feedback-controlled amplifiers we have

$$X_{N-2} = \prod_{i=1}^{N-2} \frac{K_{1h}(1+T_i p)}{T_i p + 1 + \mu_i K_{1h}} (Y_{\text{ref}} - Y_{\text{out}}), \tag{7.21}$$

$$X_1 = \frac{K_1}{D_1(p)} X_{N-2}, \tag{7.22}$$

$$X_2 = \frac{K_2}{D_2(p)} X_1 + \frac{K_2'}{D_2'(p)} f_1, \tag{7.23}$$

$$X_3 = \frac{K_3}{D_3(p)} X_2, \tag{7.24}$$

$$Y_{\text{out}} = \frac{K_4}{D_4(p)} X_3 + \frac{K_4'}{D_4'(p)} f_2. \tag{7.25}$$

Eliminating X_1, X_2, X_3, X_{N-2} between (7.21) – (7.25), we find

$$\left\{ \prod_{i=1}^{4} D_i(p) \prod_{i=1}^{N-2} (T_i p + 1 + K_{ih}\mu_i) D_2'(p) D_4'(p) + \right.$$
$$\left. + \prod_{i=1}^{4} K_i \prod_{i=1}^{N-2} K_{ih}(1 + T_i p) D_2'(p) D_4'(p) \right\} Y_{\text{out}} =$$

$$= \prod_{i=1}^{4} K_i \prod_{i=1}^{N-2} K_{ih} (1 + T_i p) D_2'(p) D_4'(p) Y_{ref} +$$

$$+ D_1(p) D_2(p) D_4'(p) \prod_{i=1}^{N-2} (T_i p + 1 + K_{ih}\mu_i) K_4 K_3 K_2' f_1 +$$

$$+ \prod_{i=1}^{4} D_i(p) D_2'(p) \prod_{i=1}^{N-2} (T_i p + 1 + K_{ih}\mu_i) f_2. \qquad (7.26)$$

The left- and the right-hand side of (7.26) are divided by $\prod_{i=1}^{N-2} K_{ih} K_1 K_2$. Assuming that K_{ih}, K_1 and K_3 are of the same order of magnitude, we put $\frac{1}{K_{hi}} = \frac{1}{K_1} = \frac{1}{K_3} = m$ and after elementary manipulations we obtained from (7.26)

$$\left\{ m^2 \prod_{i=1}^{4} D_i(p) \left[m^{N-2} \prod_{i=1}^{N-2} (1 + T_i p) + m^{N-3}\mu \sum_{j=1} \prod_{\substack{i=1 \\ i \neq j}}^{N-2} (1 + T_i p) + \right.\right.$$

$$\left. + m^{N-4}\mu^2 \sum_{j=1} \prod_{\substack{i=1 \\ i \neq j}}^{N-2} (1 + T_i p) + \ \dots \ + \mu^{N-2} \right] +$$

$$\left. + K_2 K_4 \prod_{i=1}^{N-2} (1 + T_i p) \right\} D_2'(p) D_4'(p) Y_{out} = m \left[m^{N-2} \prod_{i=1}^{N-2} (1 + T_i p) + \right.$$

$$+ m^{N-3}\mu \sum_{j=1} \prod_{\substack{i=1 \\ i \neq j}}^{N-2} (1 + T_i p) + \ \dots \ + \mu^{N-2} \Big] \times$$

$$\times K_4 K_2' D_1(p) D_2(p) D_4(p) f_1 + m^2 \left[m^{N-2} \prod_{i=1}^{N-2} (1 + T_i p) + \right.$$

$$\left. + m^{N-3}\mu \sum_{j=1} \prod_{\substack{i=1 \\ i \neq j}}^{N-2} (1 + T_i p) + \ \dots \ + \mu^{N-2} \right] \times$$

$$\times \prod_{i=1}^{4} D_i(p) D_2'(p) f_2 + K_2 K_4 \prod_{i=1}^{N-2} (1 + T_i p) D_2' D_4'(p) Y_{ref}. \qquad (7.27)$$

Here Σ indicates that the first sum is taken over the combinations of the products of all the subscripts, except $j = 1$, the next sum is taken for $j \neq 2$, etc.

We see from (7.27) that the gain of the feedback amplifiers does not affect the noise, neither enhancing nor suppressing it. The only contribution from the feedback amplifiers is that they alter the noise amplitude by a factor of M^{N-2}. If $M < 1$, the noise is appropriately amplified. As in the previous examples, the noises are suppressed by the gain of the noisefree elements without feedback. If their gain is sufficiently high, noise is effectively eliminated.

Let us now consider the left-hand side of (7.27) which, when made equal to zero, gives the characteristic equation that determines the stability (realizability) of the system. Thus,

$$m^N \prod_{i=1}^{N-2} (1 + T_i p) \prod_{i=1}^{4} D_i(p) +$$

$$+ m^{N-1}\mu \sum_{i=1}^{N-2} \prod_{\substack{i=1 \\ i \neq j}}^{N-2} (1 + T_i p) \prod_{i=1}^{4} D_i(p) + \ \dots \ + m^2 \mu^{N-2} \prod_{i=1}^{4} D_i(p) +$$

$$+ K_2 K_4 \prod_{i=1}^{N-2} (1 + T_i p) = 0. \qquad (7.28)$$

The degrees of any two adjoining polynomials in (7.28), with the exception of the last pair, differ by 1, and the difference in the degrees of the last two polynomials is 2, since the degree of the polynomial $\prod_{i=1}^{4} D_i(p)$ is by assumption N (we are dealing with aperiodic components).

The structural stability criteria formulated in Chapter Three are thus satisfied. The degenerate equation in this case is

$$K_2K_4\prod_{i=1}^{N-2}(1+T_ip)=0. \tag{7.29}$$

It always satisfies the stability conditions. In order for the system to be stable (realizable) it is clearly necessary and sufficient that the auxiliary equation of third kind satisfies the stability conditions. The auxiliary equation can be made to satisfy the stability conditions by an appropriate choice of the time constants T_i, the feedbacks, and the gain factors μ_i and K_2K_4.

We have thus proved the realizability of these structures. In this case we have incorporated in the system $N-2$ high-gain amplifiers. If $N-1$ amplifiers are introduced, we obtain an auxiliary equation of the first kind, which in our particular case always satisfies the stability conditions, as it can be reduced to the form

$$\prod_{i=1}^{n}(1+T_iq)=0. \tag{7.30}$$

The number of amplifiers in the system can be reduced to $N/2$. The corresponding auxiliary equation is of the second kind (see Chapter Three), and the feedback parameters (second-order feedback loops are used in this case) should be so chosen that the auxiliary equation satisfies the stability conditions.

§ 7.4. REALIZABILITY OF NOISEPROOF STRUCTURES IN MULTIVARIABLE CONTROL SYSTEMS

The results of the preceding section suggest a convenient approach to realizability for multivariable control systems. It is of course clear that the noise-rejecting gains should not be stabilized. The structure of the system in Figure 7.2 should therefore be so modified that the system becomes realizable and the high-gain elements K_1 and K_3 are left unstabilized. Figure 7.5 is a block diagram of an i-th subsystem which meets these requirements.

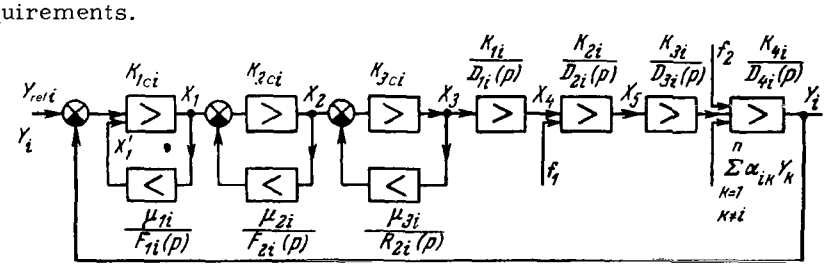

FIGURE 7.5. Illustrating realizability conditions.

Making use of the nomenclature of Figure 7.5, we obtain a set of equations describing the behavior of this system:

$$X_1 = K_{1c1}(Y_{ref\,1} - Y_1 - X_1), \tag{7.31}$$

$$X_1' = \frac{\mu_{11}}{F_{11}(p)} X_1, \tag{7.32}$$

whence

$$X_1 = K_{1c1} Y_{ref} - K_{1c1} Y_1 - \frac{\mu_{11}}{F_{11}(p)} X_1,$$

or

$$X_1\left(1 + \frac{\mu_{11} K_{c11}}{F_{11}(p)}\right) = K_{c11} Y_{ref} - K_{c11} Y_1, \tag{7.33}$$

and similarly

$$X_2\left(1 + \frac{\mu_{12} K_{c12}}{F_{12}(p)}\right) = K_{c12} X_1, \tag{7.34}$$

$$X_3\left(1 + \frac{\mu_{13} K_{c13}}{F_{13}(p)}\right) = K_{c13} X_2, \tag{7.35}$$

$$X_4 = \frac{K_{11}}{D_{11}(p)} X_3, \tag{7.36}$$

$$X_5 = \frac{K_{21}}{D_{21}(p)} X_4 + \frac{K_{21}}{D_{21}(p)} f_1, \tag{7.37}$$

$$X_6 = \frac{K_{31}}{D_{31}(p)} X_5, \tag{7.38}$$

$$Y_i = \frac{K_{41}}{D_{41}(p)} X_6 + \frac{K_{41}}{D_{41}(p)} f_2 - \frac{K_{41}}{D_{41}(p)} \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k. \tag{7.39}$$

Eliminating X_1, X_2, X_3, X_4, X_5 and X_6 between (7.33)—(7.39), we obtain after elementary manipulations the following equation for the i-th controlled variable:

$$\left[\prod_{k=1}^{4} D_{ik}(p) \prod_{k=1}^{3} (F_{ik}(p) + \mu_{ik} K_{c\,ik}) + \prod_{k=1}^{4} K_{ik} \prod_{k=1}^{3} K_{c\,ik} \prod_{k=1}^{3} F_{ik}(p)\right] Y_i +$$

$$+ K_{4i} \prod_{k=1}^{3} D_{ik}(p) \prod_{k=1}^{3} (F_{ik}(p) + \mu_{ik} K_{c\,ik}) \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik}(p) Y_k =$$

$$= \prod_{k=1}^{4} K_{ik} \prod_{k=1}^{3} K_{c\,ik} \prod_{k=1}^{3} F_{ik}(p) Y_{ref\,i} + D_{1i}(p) \prod_{k=1}^{3} (F_{ik}(p) +$$

$$+ \mu_{ik} K_{c\,ik}) \prod_{k=2}^{4} K_{ik} f_1 + K_{4i} \prod_{k=1}^{3} D_{ik}(p) \prod_{k=1}^{3} (F_{ik}(p) + \mu_i K_{c\,ik}) f_2. \tag{7.40}$$

Putting $i = 1, \ldots, n$, we obtain a complete set of equations describing this multivariable system.

Suppose that the gains K_{ci} of the feedback amplifiers and the gains K_{c1} and K_{c3} can be made sufficiently large (theoretically infinite). For the sake of simplicity we assume that the gain factors are all of the same order of magnitude. Putting

$$\frac{1}{K_{c\,i3}} = \frac{1}{K_1} = \frac{1}{K_3} = m$$

and dividing (7.40) through by $\prod_{k=1}^{3} K_{c\,k} K_1 K_3$, we obtain after simple manipulations

$$\left\{ m^2 \prod_{k=1}^{4} D_{ik}(p) \prod_{k=1}^{3} [m_k F_{ik}(p) + \mu_{ik}] + K_{i2} K_{i4} \prod_{k=1}^{3} F_{ik}(p) \right\} Y_i +$$
$$+ m^2 K_{4i} \prod_{k=1}^{3} D_{ik}(p) \prod_{k=1}^{3} [m_k F_{ik}(p) + \mu_{ik}] \sum_{\substack{k=1 \\ k \neq j}}^{n} \alpha_{ik}(p) Y_k =$$
$$= K_{i2} K_{i4} \prod_{k=1}^{3} F_{ik}(p) Y_{\text{ref}\,i} + m D_i(p) \prod_{k=1}^{3} [m_k F_{ik}(p) + \mu_{ik}] \times$$
$$\times K_{2i} K_{4i} f_1 + m^2 K_{4i} \prod_{k=1}^{3} D_i(p) \prod_{k=1}^{3} [m_k F_{ik}(p) + \mu_{ik}] f_2. \qquad (7.41)$$

From (7.41), for $m \to 0$,

$$\lim_{m \to 0} Y_i = Y_{\text{ref}\,i} \qquad (i = 1, 2, \ldots, n).$$

The realizability of this configuration is determined by the stability of the multivariable control system as a whole. We have shown previously (see Chapter Three) that a general multivariable system with infinite-gain stability can be obtained if each single-variable subsystem is stable at infinite gain; hence, to obtain the necessary realizability conditions, it is sufficient that the roots of the equation

$$m^2 \prod_{k=1}^{4} D_{ik}(p) \prod_{k=1}^{3} [m_k F_{ik}(p) + \mu_{ik}] + K_{i2} K_{i4} \prod_{k=1}^{3} F_{ik}(p) = 0 \qquad (7.42)$$

remain stable for $m \to 0$. We write equation (7.42) in expanded form:

$$\prod_{k=1}^{4} D_{ik}(p) \{ [m^5 F_{1i}(p) F_{2i}(p) F_{3i}(p) + m^4 [\mu_{2i} F_{1i}(p) F_{2i}(p) +$$
$$+ \mu_{1k} F_{i2}(p) F_{i3}(p) + \mu_{3i} F_{1i}(p) F_{i2}(p)] +$$
$$+ m^3 [\mu_{1i} \mu_{2i} F_{i3}(p) + \mu_{i2} \mu_{i3} F_{i1}(p) + \mu_{i1} \mu_{i3} F_2(p)] +$$
$$+ m^2 \mu_{1i} \mu_{2i} \mu_3]] + K_{i2} K_{i4} F_{i1}(p) F_{i2}(p) F_{i3}(p) = 0. \qquad (7.43)$$

The small parameter in braces appears in order of descending powers; the polynomials multiplying the small parameter are likewise in a descending order, and all this corresponds to structural stability for $m \to 0$. It only remains to consider the polynomial

$$m^2 \mu_{i1} \mu_{i2} \mu_{i3} \prod_{k=1}^{4} D_{ik}(p) + K_{i2} K_{i4} F_{i1}(p) F_{i2}(p) F_{i3}(p) = 0. \qquad (7.44)$$

Equation (7.44) corresponds to a realizable system if

$$d - \nu \leqslant 2, \qquad (7.45)$$

where d is the degree of the polynomial $\prod_{k=1}^{4} D_{ik}(p)$, and ν the degree of the polynomial $\prod_{k=1}^{3} F_{ik}(p)$.

It is clearly not always possible to choose such a number of feedback amplifiers that the structural conditions (7.45) are satisfied.

When (7.45) is satisfied, the necessary realizability conditions are in a sense satisfied also. The sufficient conditions are satisfied if the degenerate equation of the multivariable system and the auxiliary equations of first and second kind for the entire system satisfy the respective stability conditions. The stability conditions for the degenerate and the auxiliary equations are generally satisfied by a judicious choice of the feedback parameters F_{ik} and μ_{ik}. In any case, this does not constitute a problem. We have thus proved the property of structural noise rejection for general multivariable control systems.

§ 7.5. SELF-ADJUSTMENT PROPERTIES IN A CASE WHEN THE DISTURBANCES CAN BE DIRECTLY MEASURED

Consider an automatic control system where the plant properties (characteristics) are highly sensitive to external disturbances, which are applied directly to the plant. In this section we will deal with a case when the disturbances acting on the controlled object can be measured directly. We start with a discussion of single-variable systems, and subsequently the results will be extended to multivariable control systems.

Suppose that the automatic control system is optimal with regard to a certain quality criterion. The system parameters are calculated and chosen ignoring the action of noise, but the system drifts away from the optimum setting due to noise interference. Our problem is to alter the structure and to choose the system parameters so that the optimization attained without noise holds in the noisy case too. As we have previously remarked, it is assumed that the noises acting on the system can be measured. It thus remains to apply the results of the theory of combined systems considered in Chapter Five.

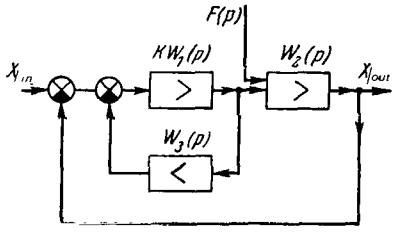

FIGURE 7.6. Illustrating realizability with the aid of real stabilizers.

Take a single-variable control system shown diagramatically in Figure 7.6. In this figure $W_2(p)$ is the transfer function of the plant, $KW_1(p)$ and $W_3(p)$ are the transfer functions of the control system and the stabilizer, $F(p)$ is the external disturbance; $KW_1(p)$ and $W_3(p)$ are so chosen that in the absence of disturbances $F(p)$ the optimum process (with regard to a certain quality criterion) is attained for a sufficiently high gain K.

For example, in automatic potentiometric control with nonlinear feedback, the optimum process is attained when the system gain is infinite /39/. The system of course should remain stable as the gain is increased.

We will prove the following proposition. For $K \to \infty$ the structure in Figure 7.6 without external disturbances is equivalent to the structure in Figure 7.7 with external disturbances. In other words, measurable external disturbances can be rejected if they are delivered as an additional signal to the stabilizer input. This proposition follows from the results of Chapter Five, and its proof is fairly obvious. Indeed, the transfer function of Figure 7.6 without external disturbances is

$$K(p) = \frac{X_{out}(p)}{X_{in}(p)} = \frac{\frac{KW_1(p)W_2(p)}{1+KW_1(p)W_3(p)}}{1+\frac{KW_1(p)W_2(p)}{1+KW_1(p)W_3(p)}} = \\ = \frac{KW_1(p)W_2(p)}{1+KW_1(p)W_3(p)+KW_1(p)W_2(p)}. \quad (7.46)$$

In the limit $K \to \infty$ we find

$$K_{out}(p) = \lim_{K\to\infty} K(p) = \frac{W_2(p)}{W_3(p)+W_2(p)}. \quad (7.47)$$

The transfer function of Figure 7.7 is obtained from the following equations:

$$Y(p) = KW_1(p)\{X_{in}(p) - X_{out}(p) - [Y(p)+F(p)]W_3(p)\}, \quad (7.48)$$

$$X_{out}(p) = W_2(p)[Y(p)+F(p)]. \quad (7.49)$$

Solving (7.48) — (7.49) for $X_{out}(p)$, we find

$$X_{out}(p) = \frac{KW_1(p)W_2(p)X_{in}(p)+W_2(p)F(p)}{1+KW_1(p)W_2(p)+KW_1(p)W_3(p)}, \quad (7.50)$$

whence

$$\lim_{K\to\infty} \frac{X_{out}(p)}{X_{in}(p)} = \frac{W_2(p)}{W_2(p)+W_3(p)}, \quad (7.51)$$

i.e., the same expression as (7.47). We see that for a sufficiently high gain, the system in Figure 7.7 behaves like an adaptive system in the sense that its characteristics remain fixed despite the presence of quite general external disturbances.

§ 7.6. CASE OF NOISY PLANT (THE DISTURBANCES CANNOT BE MEASURED)

We now consider the case of a plant whose characteristics are altered by external disturbances which are not amenable to direct measurement. This is a very common case in practice.

Let the plant parameters in the noisefree case be known. A control system is then designed for the noisefree case and optimized by indefinitely increasing the gain K. Without noise, the system has the structure shown

in Figure 7.6. We have shown in the preceding section that for $K \to \infty$ the transfer function of the single-variable control system is

$$K_{out}(p) = \frac{W_2(p)}{W_3(p) + W_2(p)}.$$

We now proceed to design the next structure (Figure 7.8). The controller output (X in Figure 7.8) is delivered to the input of the real plant and to the input of a model with a transfer function $W_2'(p)$ (this is the transfer function of the ideal, noisefree plant). The difference of output signals of the ideal and the real plant is delivered through a transducer $W_{tr}(p)$ to the stabilizer input.

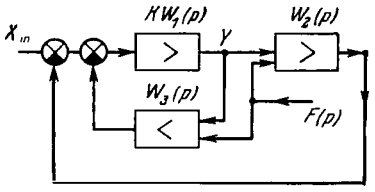

Figure 7.7. Application of stabilizers to noise rejection.

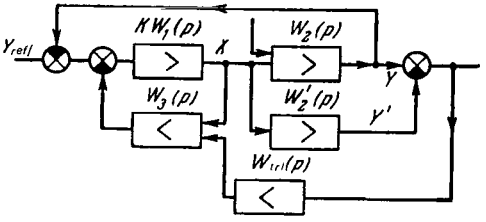

FIGURE 7.8. The use of an ideal plant.

The transfer function of the system in Figure 7.8 is obtained from the relations

$$X(p) = KW_1(p)\{Y_{ref}(p) - Y_{out}(p) - W_3(p)[X(p) + (Y_{out}(p) - Y'_{out}(p))W_{tr}(p)]\}, \quad (7.52)$$

$$Y_{out}(p) = W_2(p)[X(p) + F(p)], \quad (7.53)$$

$$Y'_{out}(p) = W'_2(p)X(p). \quad (7.54)$$

In the noisefree case $W_2(p) = W_2'(p)$, and since in (7.52) — (7.54) the disturbances are represented by a separate term, we write from the above

$$[1 + KW_1(p)W_3(p) + KW_1(p)W_2(p)]Y_{out}(p) = KW_1(p)W_2(p)Y_{ref}(p) - KW_1(p)W_2^2(p)W_{tr}(p)W_3(p)F(p) + W_2(p)F(p) + KW_1(p)W_2(p)W_3(p)F(p). \quad (7.55)$$

The stabilizer ratio is so chosen that the structure is stable at infinite gain K. Dividing (7.55) through by K, we find in the limit $K \to \infty$

$$[W_3(p) + W_2(p)]Y_{out}(p) = W_2Y_{ref}(p) + [W_2(p)W_3(p) - W_2^2(p)W_{tr}(p)W_3(p)]F(p). \quad (7.56)$$

We see from (7.56) that noise rejection is ensured if the transducer ratio satisfies the equation

$$W_2(p)W_3(p) - W_2^2(p)W_{tr}(p)W_3(p) = 0$$

or

$$W_{tr}(p) = \frac{1}{W_2(p)}. \quad (7.57)$$

A circuit with a transfer function $\frac{1}{W_2(p)}$ can be designed by the common methods used in the synthesis of structures with infinite-gain stability. The higher the gain, the closer the resultant transfer function to the sought value.

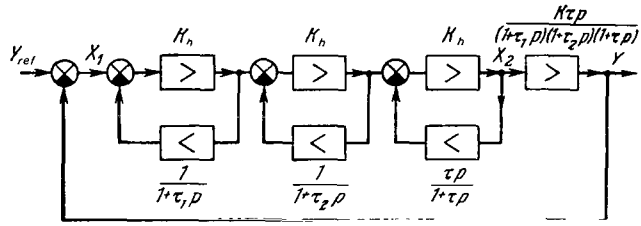

FIGURE 7.9. An example of a third-order system with variable K_h.

As an example let us consider the case of a plant with the transfer function

$$W_2(p) = \frac{K\tau p}{(1+T_1 p)(1+T_2 p)(1+\tau p)}. \tag{7.58}$$

The transducer ratio is

$$W_{tr}(p) = \frac{1}{W_2(p)} = \frac{(1+T_1 p)(1+T_2 p)(1+\tau p)}{K\tau p}. \tag{7.59}$$

Three high-gain amplifiers K_h are connected in series. They are feedback-controlled (Figure 7.9) by transfer functions $\frac{1}{1+T_1 p}$, $\frac{1}{1+T_2 p}$ and $\frac{K\tau p}{1+\tau p}$. Transfer functions of this kind can obviously be synthesized without any difficulty. To find the transfer function of the structure in Figure 7.9, we start with the equation

$$\frac{X_2(p)}{X_1(p)} = \frac{K_h}{1+\frac{K_h}{1+T_1 p}} \cdot \frac{K_h}{1+\frac{K_h}{1+T_2 p}} \cdot \frac{K_h}{1+\frac{KK_h\tau p}{1+\tau p}}. \tag{7.60}$$

Dividing the numerator and the denominator of each of the three fractional factors by K_h, we find

$$\frac{X_2(p)}{X_1(p)} = \frac{1}{\frac{1}{K_h}+\frac{1}{T_1 p+1}} \cdot \frac{1}{\frac{1}{K_h}+\frac{1}{1+T_2 p}} \cdot \frac{1}{\frac{1}{K_h}+\frac{K\tau p}{1+\tau p}}. \tag{7.61}$$

Putting $\frac{1}{K_h} = m$, we obtain after simple manipulations

$$\frac{X_2(p)}{X_1(p)} = \frac{(1+T_1 p)(1+T_2 p)(1+\tau p)}{Q_1}, \tag{7.62}$$

where

$$Q_1 = m^3(1+T_1p)(1+T_2p)(1+\tau p)+ \\ +m^2[(1+T_1p)(1+T_2p)[K\tau p+1+\tau p]]+ \\ +m[(1+\tau p)+K\tau p(1+T_1p)(1+T_2p)]+K\tau p.$$

In the limit $K_h \to \infty$ or, equivalently, $m \to 0$, we have

$$\lim_{K\to\infty}\frac{X_2(p)}{X_1(p)} = \frac{(1+T_1p)(1+T_2p)(1+\tau p)}{K\tau p} = \frac{1}{W_2(p)} = W_{tr}(p). \qquad (7.63)$$

We have thus derived the desired transfer function.

Equation (7.63) is a degenerate equation. Realizability of the structure in Figure 7.9 thus depends on the position of the roots which tend to infinity as $m \to 0$. To solve this problem, we have to draw up the auxiliary equation and to test its coefficients for stability. Since a term $K\tau p$ enters the numerator of the last feedback loop ratio, all polynomials multiplying the small parameters are of third degree and the additional conditions should therefore be checked for the following equation only·

$$m[K\tau p(1+T_1p)(1+T_2p)+(1+\tau p)]+K\tau p.$$

The small parameter raises the order of the equation by two, and the additional conditions have the form (see Chapter Three)

$$\frac{1}{T_1}+\frac{1}{T_2}>0,$$

which is naturally always true. We have thus shown that the required transducer ratio can be obtained without difficulty. In reality, K_h are not infinite: these are large but finite numbers and the transfer function $W_{tr}(p)$ is therefore realizable only to a certain accuracy, which is higher the higher the gain K_h.

Let us now consider the realizability of an extremum control system in the multivariable case. We will assume that optimization of each controlled variable corresponds to optimization of the system as a whole. It will be clear from the following that this is not always true. At the present stage we are dealing only with the case when each controlled variable can be optimized in the previously explained sense and when optimization of all the controlled variables corresponds to optimization of the system as a whole.

Now suppose that the disturbances which cannot be measured are applied to the controlled object in each variable. We thus obtain a configuration shown in Figure 7.10, which is the i-th subsystem of the multivariable control system.

The behavior of the i-th controlled variable is described by the following set of equations in Laplace transforms

$$X_i = K_{1i}W_{1i}(p)[Y_{ref}-Y_{out\ i}-Y_i'], \qquad (7.64)$$
$$Y_i' = W_{3i}(p)(X_i+X_{1i}), \qquad (7.65)$$
$$X_{1i} = W_{tr\ i}(p)Y_i, \qquad (7.66)$$
$$Y_i = Y_{out\ i}-Y_{out}', \qquad (7.67)$$
$$Y_{out}' = W_{2i}'(p)X_i, \qquad (7.68)$$

$$Y_{\text{out}\,i} = W_2(p)\left[X_i + f_i + \sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha_{ik}(p)\, Y_{\text{out}\,k}\right]. \tag{7.69}$$

Eliminating the variables Y_i', X_{1i}, X_i, Y_i, Y'_{out} between (7.64)—(7.69), we obtain

$$\begin{aligned}
&[(1 + K_{1i} W_{1i}(p)\, W_{3i}(p)\, W_{3i}(p)\, W_{\text{tr}\,i}(p) + K_{1i} W_{1i}(p)\, W_{3i}(p)]\, Y_{\text{out}\,i} \pm \\
&\pm W_{2i}(p)\, W_{3i}(p)\, W_{\text{tr}\,i}(p) \sum_{\substack{k=1 \\ k\neq i}}^{n} \alpha_{ik}(p)\, Y_{\text{out}\,k} = \\
&= K_{1i} W_{1i}(p)\, W_{3i}(p)\, Y_{\text{ref}\,i} + W_{2i}(p)\, W_{3i}(p)\, W_{\text{tr}\,i}(p)\, f_1 + \\
&+ (1 + W_{1i}(p)\, W_{3i}(p)\, K_1)\left[W_{3i}(p)\, W_{\text{tr}\,i}(p) - \frac{W_{3i}(p)}{W_{2i}(p)}\right] Y'_{\text{out}}
\end{aligned} \tag{7.70}$$

or, dividing both sides by $W_{3i}(p)$,

$$\begin{aligned}
&[(1 + K_{1i} W_{1i}(p)\, W_{3i}(p))\, W_{\text{tr}\,i}(p) + K_{1i} W_{1i}(p)]\, Y_{\text{out}} + \\
&+ W_{2i}(p)\, W_{\text{tr}\,i}(p) \sum \alpha_{ik} Y_{\text{out}\,k} = K_{1i} W_{1i}(p)\, Y_{\text{ref}\,i} + W_{2i}(p)\, W_{\text{tr}\,i}(p)\, f_1 + \\
&+ (1 + W_{3i}(p)\, W_{1i}(p)\, K_1)\left[W_{\text{tr}\,i}(p) - \frac{1}{W_{2i}(p)}\right] Y'_{\text{out}}.
\end{aligned} \tag{7.71}$$

Putting $i = 1, 2, \ldots, n$, we obtain a complete set of equations describing the multivariable control system.

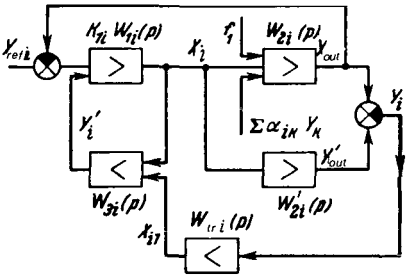

FIGURE 7.10. The use of an ideal plant.

As we have previously shown the system can be optimized in relation to each controlled variable; the disturbances f_i together with the extraneous outputs provide the noise which interferes with the given variable. The optimum is thus attained if the extraneous outputs $Y_{\text{out}\,k}$ and the disturbances f_i are successfully eliminated. We see from (7.71) that noise rejection can be achieved if

(a) $W_{\text{tr}\,i}(p) = \dfrac{1}{W_{2i}(p)}$

and

(b) $K_i \to \infty$, the system of course remaining stable as a whole.

Making use of conditions (a) and (b), we find from (7.71)

$$\left[\frac{W_{1i}(p)\, W_{3i}(p)}{W_{2i}(p)} + W_{1i}(p)\right] Y_{\text{out}\,i} = W_{1i}(p)\, Y_{\text{ref}\,i},$$

or

$$[W_{1i}(p)\,W_{3i}(p)+W_{2i}(p)\,W_{1i}(p)]\,Y_{\text{out}\,i}=W_{1i}(p)\,W_{2i}(p)\,Y_{\text{ref}\,i}, \qquad (7.72)$$

or

$$\frac{Y_{\text{out}\,i}}{Y_{\text{ref}\,i}}=\frac{W_{2i}(p)}{W_{2i}(p)+W_{3i}(p)}.$$

An equation of this kind was derived in our previous analysis of optimization of a single-variable system. This single-variable case is of some interest as a variant of multivariable systems where noninteraction is attained as a byproduct of self-optimization of each subsystem, which by assumption corresponds to optimization of the system as a whole. In point of fact, such self-optimization is feasible only if noninteraction and invariance are ensured simultaneously.

§ 7.7. DISTURBANCES APPLIED AT VARIOUS POINTS OF THE FORWARD PATH AND IN THE FEEDBACK PATH

Consider the case when the disturbances are injected at various points along the forward path, with the exception of the input, and also along the feedback path. This case is illustrated in Figure 7.11. The results of § 7.2 clearly suggest that by increasing the gain of the forward path one can reject all the noises acting in that path and compensate the contribution from the extraneous controlled variables. Now, if the plant characteristics are altered in response to these disturbances, the resulting structure is equivalent in its properties to an adaptive system. The unsolved problem is noise rejection in the feedback path, but here we can apply the conditions of structural noise rejection derived in Sec. 7.2.

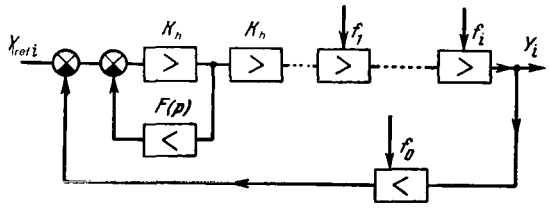

FIGURE 7.11. Noise in forward and feedback paths

Structural noise rejection can be attained as follows (Figure 7.12). An amplifier is connected in the feedback path, immediately after the output; its gain can be made sufficiently large (theoretically infinite). Another amplifier with gain K'_{2h} close to zero is connected after the noisy feedback element, so that

$$K_{1h}\frac{1}{K_{2h}}=1.$$

Simple calculations show that $K_{1h} \to \infty$, $K_{2h} \to 0$, and $K_{1h}K_{2h} = 1$ the noise in the feedback path is effectively suppressed.

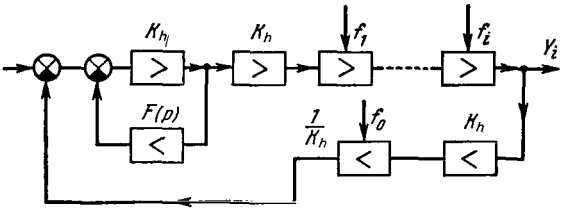

FIGURE 7.12. Illustrating noise rejection in the feedback path.

In conclusion of this section let us review the results of §§ 7.2 and 7.6. We have dealt there with noisefree input signals. The noise rejection techniques have been essentially developed for cases when the noises are not amenable to direct measurements. The use of the ideal plant in § 7.5 enabled us, besides synthesis of $W_{ir}(p)$, to ensure stabilization with the aid of simple passive circuits, whereas the method presented in this section requires special amplifiers that realize sufficiently ideal derivatives.

The system properties can also be improved in the case of noisy input, and this possibility is actually considered in § 7.5. Indeed if noise rejection follows the method of § 7.2, an increase in gain enhances the noise which is delivered to the input together with the reference signal. From this point of view, if noise is injected together with the reference signal, the suppression of all other noises that incidentally enter the system requires unambiguous isolation of the original noise, and this is an obvious shortcoming of the method. If an ideal plant is used, and especially if the spectral composition of noise is different from the spectral composition of the reference signal, the parameters of the stabilizer $W_3(p)$ can be chosen so as to minimize the input noise. When the input is a mixture of the reference signal $Y_{ref\,i}$ and noise $f_{in\,i}$, calculations along the same lines as before give the following expression for the output in a system using an ideal plant:

$$Y_{out\ i} = \frac{W_2(p)}{W_3(p) + W_2(p)} Y_{ref\,i} + \frac{W_2(p)}{W_3(p) + W_2(p)} f_{in}. \qquad (7.73)$$

If $W_3(p)$ is appropriately chosen and the spectral composition of the disturbance is taken into consideration, the contribution from the second term in (7.73) can be minimized.

§ 7.8. SOME ADDITIONAL TOPICS AND ESTIMATES

In the preceding discussion, a realizable structure was one that remained stable at infinite gain. The concept of realizability used in current literature has a broader sense, and our analysis should be correspondingly augmented. Moreover, when asymptotic methods are applied

(in our case the asymptotic behavior constitutes transition to the limit $m \to 0$ or $K \to \infty$, one always has to consider to what extent the theoretical results are applicable in practice, when the coefficients after all remain finite, and what errors are incurred in the asymptotic approximation. We therefore first have to consider the following problems.

1. In control systems (as in any dynamic system), there are always some parasitic, spurious parameters which at high gain may have a marked influence on system dynamics. Two aspects of this question should be considered:

(a) Will small parameters have a marked influence on system behavior if they can be made quantitatively as small as desired?*

(b) How are we to determine the quantitative effect of small but finite parasitic parameters on system dynamics?

2. In real systems the gain cannot be made arbitrarily large: it may be raised to a certain large but nevertheless finite value. What constitutes "sufficiently large" gain, or in other words what are the numerical values of gain for which the results obtained assuming infinite gain are applicable?

3. What is the effect of certain kinds of nonlinearity on system behavior?

We solve these problems by following the same procedure as before: first we consider single-variable control system and then generalize the results to multivariable configurations.

1. Quantitative estimation of small parameters

Let an automatic control system be described by an N-th order differential equation. Moreover suppose that the system incorporates n small parameters, each increasing the order of the equation by one. The characteristic equation that corresponds to the degenerate differential equation obtained when the n small parameters are ignored is an algebraic equation of $(N-n)$-th degree, and its general form is

$$a_0 p^{N-n} + a_1 p^{N-n-1} + \ldots + a_{N-n} = 0. \tag{7.74}$$

Let the roots of equation (7.74) be z_i, $i = 1, 2, \ldots, N-n$. Then

$$a_0 z_i^{N-n} + a_1 z^{N-n-1} + \ldots + a_{N-n} = 0. \tag{7.75}$$

Introduction of the small parameters has a twofold effect. First, the roots of the degenerate equation (7.75) are altered; second, n new roots are added, which tend to infinity as the small parameters approach zero.

Let the introduction of small parameters alter the i-th root of the degenerate equation by Δz_i. The coefficients of the degenerate equation acquire corresponding increments Δa_i and N new terms with the coefficients Δb_i appear in the equation. The complete characteristic equation with small parameters is thus written in the form

$$\begin{aligned}\Delta b_0 (z_i + \Delta z_i)^N + \Delta b_1 (z_i + \Delta z_i)^{N-1} + \ldots + (a_0 + \Delta a_0)(z_i + \Delta z_i)^{N-n} + \\ + (a_1 + \Delta a_1)(z_i + \Delta z_i)^{N-n-1} + \ldots + a_{N-n} + \Delta a_{N-n} = 0.\end{aligned} \tag{7.76}$$

* It is here that we encounter the problem of coarse and noncoarse systems (in the sense of A.A. Andronov) in all its acuteness.

Expanding, we obtain

$$\Delta b_0\left[z_i^N + Nz_i^{N-1}\Delta z_i + \frac{N(N-1)}{2!}z_i^{N-2}\Delta z_i^2 + \ldots\right] +$$
$$+ \Delta b_1\left[z_i^{N-1} + (N-1)z_i^{n-2}\Delta z_i + \frac{(N-1)(N-2)}{2!}z_i^{N-3}\Delta z_i^2 + \ldots\right] + \ldots$$
$$\ldots + a_0 z_i^{N-n} + a_0(N-n)z_i^{N-n-1}\Delta z_i + \ldots + \Delta a_0 z_i^{N-n} +$$
$$+ \Delta a_1(N-n)z_i^{N-n-1}\Delta z_i + \ldots + a_{N-n} + \Delta a_{N-n} = 0. \qquad (7.77)$$

Making use of (7.75) and ignoring terms of second and higher orders of smallness, we obtain

$$\Delta b_0 z_i^N + \Delta b_i z_i^{N-1} + \ldots + a_0 z_i^{N-n} +$$
$$+ \Delta a_1 z_i^{N-n-1} + \ldots + \Delta a_{N-n} = \varphi'(z)|_{z=z_i}\Delta z_i, \qquad (7.78)$$

where $\varphi'(z)|_{z=z_i}$ is the derivative with respect to z of equation (7.75) at $z = z_i$.

From (7.78) we obtain

$$\Delta z_i = \frac{\Delta b_0 z_i^N + \Delta b_1 z_i^{N-1} + \ldots + \Delta a_0 z_i^{N-n} + \Delta a_1 z_i^{N-n-1} + \ldots + \Delta a_{N-n}}{\varphi'(z)_{z=z_i}}. \qquad (7.79)$$

This expression relates the root increment to the increments of the coefficients to terms of second order of smallness.

If the numerical values of the small parameters and the roots of the degenerate equation are known, the error in the roots calculated from the degenerate characteristic equation can be found using relation (7.79). If the roots of the degenerate equation are real, relation (7.79) gives the error in the decrements of damping; if the roots are complex, relation (7.79) simultaneously gives the error in the damping decrements and in the free oscillation frequencies of the system.

The problem can be approached differently. Let the permissible error be known (e.g., in percent of the root of the degenerate equation); our aim is to find such numerical values of the small parameters that the error incurred when these parameters are omitted does not exceed the permissible error.

Let the permissible error be ε, so that

$$\Delta z_i = \varepsilon z_i.$$

The numerator in the right-hand side of (7.79) is a known function of the small parameters m. Putting $f(m)$ for this function, we rewrite equation (7.79) as

$$f(m_0) = \varepsilon z_i \varphi'(z)|_{z=z_i}. \qquad (7.80)$$

Here m_0 is the largest of the small parameters.

The error naturally does not exceed the permissible value ε if, for any m,

$$m < m_0. \qquad (7.81)$$

In most cases the effect of the small parameters on system dynamics can be quantitatively determined by considering only the errors in the roots of the degenerate equation, although strictly speaking the change in the

position of the roots generated by the small parameters should also be estimated.

We have previously shown that when the small parameters approach zero, the roots obtained from the auxiliary equation tend to infinity. In reality, however, the small parameters are finite quantities, and the corresponding roots are therefore located not at infinity but at some finite distance from the origin.

For purposes of evaluation of the transient process via the degenerate equation it is desirable to have the real roots generated by the small parameters considerably farther to the left from the imaginary axis than the leftmost root of the degenerate equation; alternatively, the absolute value of the complex root calculated with allowance for the small parameters should be substantially greater than the absolute value of the complex root of the degenerate equation. Then, all other conditions being equal, the transient components contributed by the small parameters will have a negligible influence on the overall control curve.

If there are n small parameters, the auxiliary equation for $m \to 0$ has the form

$$C_0^n + C_1 q^{n-1} + \ldots + C_n = 0. \tag{7.82}$$

The small parameters modify (7.82) as follows:

$$(C_0 + \Delta C_0) q^n + (C + \Delta C_1) q^{n-1} + \ldots + C_n + \Delta C_n = 0. \tag{7.83}$$

Proceeding along the same lines as before, we obtain an approximate expression for the error in the i-th root due to the small parameters·

$$\Delta q_i = \frac{\Delta b_0 q^N + \Delta b_1 q^{N-1} + \ldots + \Delta a_0 q^{N-n} + \ldots + \Delta a_{N-n}}{\varphi'(q)_{q=q_i}}, \tag{7.84}$$

where $\varphi'(q)|_{q=q_i}$ is the derivative with respect to q of (7.82) for $q=q_i$. If q_i is known, the error in the root can be found. The actual value of the root is given by the equality

$$p_i = \frac{q_i + \Delta q_i}{m}. \tag{7.85}$$

The relations of this section are suitable for the determination of the numerical values of small parameters which raise the degree of the equation at most by one. For certain structures the small parameter is the reciprocal gain of the stabilized section of the control loop, and our relations can thus be applied to determine the gain. Seeing that

$$m = \frac{1}{K_{st}}, \tag{7.86}$$

we write (7.85) in the form

$$p_i = (q_i + \Delta q_i) K_{st}. \tag{7.87}$$

If K_{st} is known, the true value of the root can be found from (7.87). The reverse procedure is more convenient in practice: first find the roots of the auxiliary equation with $m = 0$, then assuming a certain permissible error Δq use (7.87) to determine the gain K_{st}.

2. Coarseness in the sense of A.A. Andronov

Let each closed-loop subsystem of the control system have a certain number of small parameters (in general, different subsystems need not have the same number of small parameters). The starting set of equations can be written in the form

$$\left.\begin{aligned}
&[D_1(p)M_1(p)F_1(mp)+K_1]X_1=-K_1'M_1(p)X_0,\\
&[D_2(p)M_2(p)F_2(mp)+K_2]X_2=-K_2'M_2(p)X_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&[D_n(p)M_n(p)F_n(mp)+K_n]X_n=-K_n'M_n(p)X_{n-1}.
\end{aligned}\right\} \qquad (7.88)$$

Here $F_i(mp)$ is a polynomial whose coefficients are functions of the small parameters m; $D_i(p)$, $M_i(p)$, K_i and K_i' are the operators and gain factors of the controlled object and the controller in various closed-loop subsystems.

Suppose that the parasitic parameters are the time constants of the serially connected aperiodic elements in the loop. Then

$$F_i(mp)=m^{\rho}F_{\rho}(p)+m^{\rho-1}F_{\rho-1}(p)+\ \ldots, \qquad (7.89)$$

where ρ is determined by the number of small parameters introduced.

Expression (7.89) is quite general, provided that each small parameter increases the degree of the equation at most by one.

In this case, however, the degree of the general characteristic equation increases by an amount which is equal to the number of small parameters introduced. The system is stable for $m\to 0$ if the auxiliary, as well as the degenerate, equation satisfies the stability conditions. If the parasitic small parameters enter the system in such a way that they are equivalent to a chain comprising an appropriate number of aperiodic elements connected in series, the stability conditions are automatically satisfied for small m. In general, the small parasitic time constants can always be so adjusted that, if sufficiently small, they will not affect the stability of the system. Hence it follows that systems belonging to this class are coarse in A.A. Andronov's sense.

§ 7.9. DETERMINATION OF GAIN

The method developed in the previous section for the determination of small m and high K is universally applicable only in those cases when the small parameters raise the degree of the equation at most by one.

Before this method can be applied, the roots of the degenerate and the auxiliary equation should be found. Determination of roots, even those of the degenerate equation, often involves considerable difficulties, since the equation may be of a fairly high degree. Therefore, as a supplement to the general method, which is quite useful if the effect of small time constants on system dynamics is to be found, we describe in this section some methods for the determination of gain in a number of practically significant cases. We also consider the permissible margin of variation of this gain for which the previously derived rules of structure synthesis hold true.

1. Gain entering linearly the characteristic equation

In the simplest structures which retain their stability at high gain, the gain, which may vary between wide limits without causing instability, is a linear component in the equation. A specimen structure of this kind is shown in Figure 7.13.

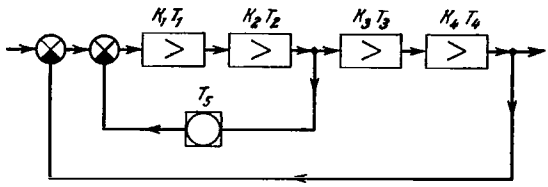

FIGURE 7.13. A structure with a real stabilizer.

K_1 and K_2 are the gains that can be varied between wide limits. The characteristic equation is

$$\prod_{i=1}^{4}(1+T_ip)(1+\tau p)+K_1K_2[\tau p(1+T_3p)(1+T_4p)+ +K_3K_4(1+\tau p)]=0. \qquad (7.90)$$

The limits of K_1K_2 for which the system remains stable can be found without difficulty. If all the other parameters are known, we plot the D-decomposition curve in the K_1K_2 plane. The equation of the D-decomposition curve in this case is

$$\overline{K_1K_2}=-\frac{\prod_{i=1}^{4}(1+T_ij\omega)(1+\tau j\omega)}{\tau j\omega(1+T_3j\omega)(1+T_4j\omega)+K_3K_4(1+\overline{\tau j\omega})}. \qquad (7.91)$$

The curve plotted from equation (7.91) is shown in Figure 7.14. The numerical values of the parameters are listed in Table 7.1.

TABLE 7.1.

T_1, sec	T_2, sec	T_3, sec	T_4, sec	τ, sec	K_2K_4
0.1	0.01	0.34	0.1	0.5	15

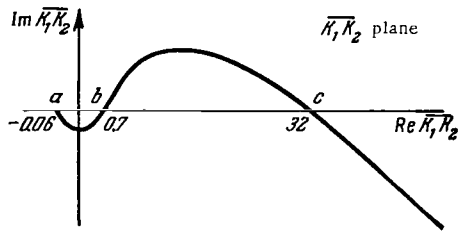

FIGURE 7.14. Illustrating the determination of K_1K_2 limits.

We see from Figure 7.14 that the system is stable in two K_1K_2 intervals: the first interval extends from $K_1K_2 = -0.6$ to $+0.7$ and the second from $K_1K_2 = 32$ to $+\infty$. The second range determines the infinite-gain stability of the system. The least value for which all the preceding conclusions hold true is obviously $K_1K_2 = 32$. In general, the characteristic equation for the structures considered in this subsection can be written in the form

$$Q(p) = K_{st} R(p) = 0, \tag{7.92}$$

and the crossover values of K_{st} are determined by examining the stable regions of the D-decomposition curve

$$\bar{K}_{st} = -\frac{Q(j\omega)}{R(j\omega)}. \tag{7.93}$$

One of the stable regions of necessity extends to infinity.

2. Gain entering nonlinearly the characteristic equation

Quantitative estimation of gain in this case is a much more complicated undertaking, especially if we are interested in the whole range and not in some single gain value from the stability region. The problem will be solved in application to structures with infinite-gain stability.

Since the system remains stable as the gain is increased, there exists a whole range of gain values where the system is stable. If the high gains are replaced by their reciprocals, we obtain a certain region of small quantities where the system is stable. This transformation will be found useful in what follows.

We have previously shown (see Chapter Three) that the characteristic equation of this class of structures with n high gains can be written in the form

$$m^n F_{N_0}(p) + m^{n-1} F_{N_1}(p) + m^{n-2} F_{N_2}(p) + \ldots \\ \ldots + m F_{N_{n-1}}(p) + F_{N_n}(p) = 0, \tag{7.94}$$

where $m = 1/K$. If the high gain parameters are not equal numerically, the characteristic equation is nevertheless written in the form (7.94), but the coefficients of the polynomials depend on some coefficients η_i which express the relationship between K_1 and K_i.

The equation of the D-decomposition curve for a sequence of descending powers of the small parameters m can be written as

$$m^n = -\frac{F_{N_1}(j\omega)}{F_{N_0}(j\omega)}\left\{m^{n-1} + \frac{F_{N_2}(j\omega)}{F_{N_1}(j\omega)}\left\{m^{n-2} + \frac{F_{N_3}(j\omega)}{F_{N_2}(j\omega)}\left\{m^{n-3} + \ldots\right.\right.\right. \\ \ldots + \left\{m + \frac{F_{N_n}(j\omega)}{F_{N_{n-1}}(j\omega)}\right\}_{n-1}\Big\}\ldots \tag{7.95}$$

We see that the small parameter to the i-th power is followed by the equation of the D-decomposition curve in the plane of that parameter,

provided that all the other small parameters and their D-decomposition equations are ignored. Putting

$$m^i = -\frac{F_{N_{i+1}}(j\omega)}{F_{N_i}(j\omega)} = D_i(j\omega), \tag{7.96}$$

we rewrite equation (7.95) in the form

$$m^n = D_n(j\omega)\{m^{n-1} - D_{n-1}(j\omega)\{m^{n-2} - D_{n-2}(j\omega)\{m^{n-3} - \\ - D_{n-3}(j\omega) \ldots \{m - D(j\omega)\}. \tag{7.97}$$

We have thus obtained an equation from which the limiting values of the reciprocal gain can be determined.

3. Initial conditions

We have previously shown that for sufficiently high gain of the stabilized elements, the transient is fully described by the degenerate equation. These results were obtained assuming zero initial conditions for the transient. In what follows we will show that the same conclusion is applicable in the general case of nonzero initial conditions. (The problem of initial conditions is moreover important because the results can be applied when the system performance is assessed in terms of the degree of stability. *)

Let the initial conditions be

$$x(t)|_{t=0} = x_0 \text{ and } x^{(i)}(t)|_{t=0} = 0 \quad (i = 1, 2, \ldots N-1). \tag{7.98}$$

The roots of the characteristic equation are designated $z_1, z_2, \ldots, z_n$.

Since the system is stable, z_i is either a negative real number or a complex number with a negative real part. The free transient component is expressed by the equation

$$x(t) = \sum_{i=1}^{N} A_i e^{z_i t}, \tag{7.99}$$

where A_i are integration constants.

To determine the N integration constants, we draw up N equations for the N initial conditions. Making use of (7.98), we obtain from (7.98) for $t = 0$

$$\left.\begin{aligned} A_1 + A_2 + \ldots + A_N &= X_0, \\ z_1 A_1 + z_2 A_2 + \ldots + z_N A_N &= 0, \\ z_1^2 A + z_2^2 A_2 + \ldots + z_N^2 A_N &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z_1^{N-1} A_1 + z_1^{N-1} A_2 + \ldots + z_1^{N-1} A_N &= 0. \end{aligned}\right\} \tag{7.100}$$

* Tsypkin, Ya.Z. and P.V. Bromberg. O stepeni ustoichivosti lineinykh sistem (Degree of Stability of Linear Systems). — Izvestiya AN SSSR, tech. sci. div., No. 12. 1945.

The determinant of (7.100) is

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix}. \tag{7.101}$$

The i-th integration constant is thus given by the equality

$$A_i = \frac{\Delta_i}{\Delta}; \tag{7.102}$$

Δ_i is the determinant (7.101) where the i-th column has been replaced by the right-hand sides of equations (7.100).

Suppose that the system has n small parameters, each increasing by one the degree of the equation. Then n out of the total of N roots recede to infinity as the n small parameters approach zero, and the other $N-n$ roots remain finite (in the limit they are equal to the $N-n$ roots of the degenerate equation).

The Laplace theorem is now applied to expand the determinants Δ and Δ_i in minors of $(N-n)$-th degree. Allowing n roots to recede to infinity, we see that

$$\lim_{z_j \to \infty} A_i' = A_i, \tag{7.103}$$

where

$$i = 1, 2, \dots, N-n; \; j = N-n+1, \dots, N$$

and

$$\lim_{z_j \to \infty} A_j' = 0, \tag{7.104}$$

where A_i' is the integration constant determined from the complete equation, A_i the corresponding constant determined from the degenerate equation.

Thus, when the roots generated by the small parameters become sufficiently large, the integration constants obtained from the degenerate equation are sufficiently close to the corresponding integration constants determined from the complete equation; the other integration constants approach zero.

Hence it follows that when the ignored parameters are sufficiently small, the transient derived from the degenerate equation is sufficiently close to the transient derived from the complete equation.

Consider a different set of initial conditions:

$$x(t)|_{t=0} = x_0, \; x^i(t)|_{t=0} = x_0^i. \tag{7.105}$$

The preceding results are fully applicable in this case too, provided that A_i and all A_j are finite. This can be easily verified by direct computation following the procedure outlined above.

§ 7.10. THE EFFECT OF SOME NONLINEARITIES

The preceding results are fully justified for systems described by linear differential equations with constant coefficients. As we have remarked in the previous section, the gain is always finite in practice and the parasitic time constants can be made arbitrarily small. The evaluation methods developed above could have applied in this case, but unfortunately, in real systems, some elements may be nonlinear. We consider here some kinds of nonlinearity and try to establish how they affect the structural properties of systems of this class.

For the sake of simplicity, we again start with the analysis of a single-variable system. The results are then readily extended to multivariable control systems.

The nonlinearity considered in this section is such that $\frac{dx_2}{dx_1} > 0$ at all points of the steady-state characteristic (x_1 is the input and x_2 is the output). A suitable example of these nonlinearities is provided by magnetization curves of electric motors, and other similar characteristics.

We will also consider the effect of these nonlinearities on the dynamics of systems with nonlinear stabilized elements. It will be assumed that the gain of the closed loop formed by the nonlinear element and the stabilizer may vary between wide limits.

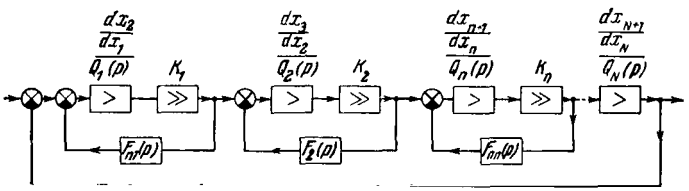

FIGURE 7.15. Estimating the effect of nonlinearities.

Figure 7.15 is a block diagram of an N-element control system with n nonlinear elements whose steady-state characteristics satisfy the conditions $\frac{dX_2}{dX_1} > 0$. A linear amplifier is connected in series with each nonlinear element, and each pair of this kind is embraced by a stabilizer $F_{n_i}(p)$. The resulting structure is stable at infinite gain.

The equation of a single loop comprising a nonlinear element, an amplifier, and a stabilizer has the form (see Figure 7.15).

$$Q_i(p)\, X_{i+1} = K \frac{dX_{i+1}}{dX_i} [X_i - F_{n_i}(p)\, X_{i+1}]$$

or

$$\left[Q_i(p) + K_i \frac{dx_{i+1}}{dx} F_{n_i}(p)\right] X_{i+1} = K_i \frac{dX_{i+1}}{dX_i} X_i, \qquad (7.106)$$

where $p=\frac{d}{dt}$. Here K_i is the gain of the linear amplifier, $\frac{dX_{i+1}}{dX_i}$ the gain of the nonlinear element, $F_{n_i}(p)$ derivatives from first to (q_i-2)-th order, where q_i is the degree of the self-operator $Q_i(p)$ of the stabilized element.

Dividing (7.106) through by $K_i\frac{dX_{i+1}}{dX_i}$ and putting $\frac{1}{K_i\frac{dX_{i+1}}{dX_i}}=m_i$, we write (7.106) in the form

$$[m_iQ_i(p)+F_{n_i}(p)]\,X_{i+1}=X_i. \tag{7.107}$$

Here m is a variable and its value is determined by the position of the element's working point on the nonlinear characteristic.

If the number of nonlinear elements is n and the total number of elements N, the equation for the N-th controlled variable is

$$\prod_{i=1}^{n}[m_iQ_i(p)+F_{n_i}(p)]\prod_{i=n+1}^{N}[Q_j(p)+K_j]\,X_N=\prod_{j=n+1}^{N}K_jX_0. \tag{7.108}$$

The characteristic equation generated by (7.108) satisfies the stability conditions for $m_i\to 0$ (or, equivalently, $K_i\to\infty$) if and only if the degenerate characteristic equation

$$\prod_{i=1}^{n}F_{n_i}(p)\prod_{j=n+1}^{N}[Q_j(p)+K_j]=0$$

and the auxiliary equation of first or second kind satisfy the stability conditions.

If these conditions are satisfied, then for sufficiently small $m\leqslant m_0$ the transient is fully determined by the degenerate equation

$$\prod_{i=1}^{n}F_{n_i}(p)\prod_{j=n+1}^{N}[Q_i(p)+K_j]\,X_N=\prod_{j=n+1}^{N}K_jX_0. \tag{7.109}$$

Thus, if the gain can be made sufficiently large, so that $\frac{1}{K_i\frac{dX_{i+1}}{dX_i}}=m\leqslant m_0$, nonlinearity of the kind being considered will have virtually no influence on the process.

We have established that the system is necessarily stable in the small. Since for nonlinearities of this kind the equivalent gain /39/ is represented by a real segment, the gain K_i and the stabilizer parameters can be so chosen that the gain-phase plot does not intersect with this segment, or else the intersection is at very low gains and the system can be regarded as stable on the whole.

Under these conditions it only remains to find the numerical value of m, and as long as $m_i<m_0$, the nonlinearity can be ignored.

Stability at high K_i is ensured by introducing ideal derivatives of up to $(q-2)$-th order. As we have already remarked, the generation of these derivatives often involves considerable technical difficulties. Instead of ideal derivatives one therefore normally uses stabilizers with a transfer function $\frac{\tau p}{1+\tau p}$. We thus proceed to consider the effect of nonlinearity on the dynamics of systems with $\frac{\tau p}{1+\tau p}$ stabilizers. The self-operators $Q_i(p)$

in this case are at most of second degree. The equation of the i-th element is

$$Q_i(p)X_{i+1}=K_i\frac{dX_{i+1}}{dX_i}(X_i-X_i'), \tag{7.110}$$

where X_i' is the stabilizer output.

The stabilizer equation is

$$(1+\tau p)X_i'=\tau pX_{i+1}. \tag{7.111}$$

Eliminating X_i' between (7.110) and (7.111), we find

$$\left[Q_i(p)(1+\tau p)+K_i\frac{dX_{i+1}}{dX_i}\tau p\right]X_{i+1}=K_i(1+\tau p)\frac{dX_{i+1}}{dX_i}X_i. \tag{7.112}$$

Differentiating in the right-hand side of this equation $p=\frac{d}{dt}$, we find

$$\left[Q_i(p)(1+\tau p)+K_i\frac{dX_{i+1}}{dX_i}\tau p\right]X_{i+1}=K_i\frac{dX_{i+1}}{dX}(1+\tau p)X_i+K_i\tau X_i\frac{d}{dt}\left(\frac{dX_{i+1}}{dX_i}\right),$$

or

$$\left[Q_i(p)(1+\tau p)+K_i\frac{dX_{i+1}}{dX_i}\tau p\right]X_{i+1}=\left[K_i\frac{dX_{i+1}}{dX_i}(1+\tau p)+K_i\tau p\frac{d}{dt}\left(\frac{dX_{i+1}}{dX_i}\right)\right]X_i. \tag{7.113}$$

Dividing (7.113) through by $K_i\frac{dX_{i+1}}{dX_i}$ and putting as before

$$\frac{1}{K_i\frac{dX_{i+1}}{dX_i}}=m_i,$$

we find

$$[m_iQ_i(p)(1+\tau_ip)+\tau_ip]X_{i+1}=(1+\tau p)X_i+\frac{\tau_i}{\frac{dX_{i+1}}{dX_i}}\frac{d}{dt}\left(\frac{dX_{i+1}}{dX_i}\right)X_i. \tag{7.114}$$

In distinction from the case of stabilization via ideal derivatives, the right-hand side of equation (7.114) contains a linear term $(1+\tau p)X_i$, which does not add to our difficulties, and a term dependent on $\frac{dX_{i+1}}{dX_i}$. The nonlinear effect cannot be assessed unless the last term in (7.114) is estimated.

Figure 7.16 is the free-running characteristic of an electric motor; this is a typical plot of nonlinearities with which we are concerned. Over sections ab and cd we have $\frac{dX_{i+1}}{dX_i}=\text{const}$, and when the working point of the nonlinear element is situated on these sections of the characteristic, the last term in (7.114) vanishes. It remains to consider the case when the working point is on the section bc. If the characteristic is smooth and well-behaved over this section (as is the case for most real elements), the last term may again be ignored, since it will slightly alter the coefficients of the equation without changing its degree. Thus, if the

gain of the linear amplifier is sufficiently high, the nonlinearity virtually does not affect the dynamics of the system.

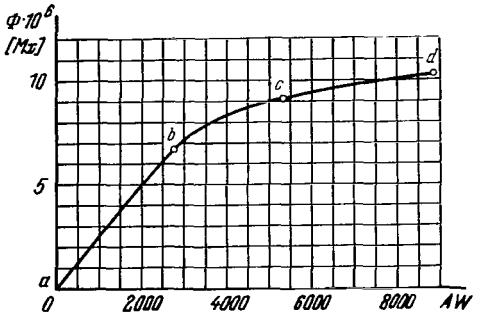

FIGURE 7.16. A saturation characteristic.

Strictly speaking, the system should have been tested for stability in the large. V. M. Popov's method /20/ provides a logical approach to this problem. However, there is no need to proceed with the general test for the very simple reason that the gain-phase plot of the open-loop system, which is needed for testing the stability in the large by Popov's method, cannot be constructed unless the numerical values of the system parameters are known. A qualitative gain-phase diagram does not yield any additional information, since in this class of structures the linear part may have a virtually arbitrary characteristic.

In this section, as in § 7.8, we have dealt with single-variable systems, but the results are readily extended to the multivariable case. An example of this generalization is provided by the preceding analysis of an n-loop system with nonlinearities.

§ 7.11. SYSTEMS WITH A RELAY ELEMENT

The use of relay elements in control circuits is of considerable interest for some problems discussed in this book. To avoid any misunderstanding we wish to stress that this is not an exposition of the theory of relay systems. Our interest in elements with a relay characteristic is due at least to three factors. First, the relay element has an infinite gain when the deviations are sufficiently small. In this sense any amplifier with an arbitrarily large gain and a zero-slope characteristic in the saturation zone can be simulated by a relay element and, conversely, a relay element can be replaced by an amplifier with such a characteristic. Second, it has been demonstrated in a number of studies on optimum control (see, e.g., /54/) that an element with a relay characteristic is an indispensable component of optimum control systems and as such of considerable interest in our analysis. Third, the sliding action of a relay system is a fundamental operating mode of the entire class of so-called variable-structure systems /8/, which have recently

become quite popular in the theory of automatic control. It has been shown in the literature /8, 71/ and will be demonstrated in the following that a sliding-action relay system is equivalent in some of its properties to an infinite-gain system.

We will consider two operating modes of systems with relay elements: (1) stable equilibrium, and (2) sliding regime.

1. Stability of relay systems

Here we are concerned with the stability of equilibrium under small deviations from the steady-state value (stability in the small). Stability as such is interpreted in the conventional sense.

The relay characteristics depicted in Figure 7.17 show that an equilibrium position which includes $x(t)=0$ is obtained both in the case shown in Figure 7.17a, where the equilibrium point is 0, and in that shown in Figure 7.17b, where

$$x-\varkappa_0<x<x+\varkappa_0.$$

Here $x(t)$ is the input signal of the relay element.

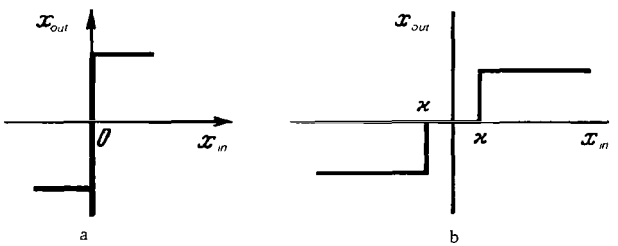

FIGURE 7.17. Relay characteristics:

(a) an ideal relay, (b) a relay with an insensitive zone.

We are particularly interested in the case $x=0$ (Figure 7.17a). The stability of blind-zone relay systems (Figure 7.17b) is determined by the linear part of the system*, since for $x(0)-\varkappa<x\leqslant x(0)<x(0)+\varkappa$ the relay element does not affect the linear part and the entire configuration behaves as an open-loop system.

The analysis of stability will be based on the characteristic equation of the linearized system. Figure 7.18 is a block diagram of a relay control system. The entire linear part is represented by a single block with a certain transfer function, and the relay element is depicted separately. It is implied that the linear part of the system is structurally representable as a one-loop circuit without local (internal) feedback. The equation of the linear part is found without difficulty. Let us consider the relay

* Tsypkin, Ya.Z. Teoriya releinykh sistem avtomaticheskogo regulirovaniya (Theory of Relay Control Systems). — Gostekhizdat. 1955.

element in some detail. The relay characteristic (Figure 7.17) is discontinuous and nondifferentiable at the origin. The very applicability of the method of small deviations and of the variational equation therefore requires special proof. The variational equation can be shown to apply with the aid of the results derived by Pontryagin and Boltyanskii /53/; physical arguments will be found in Tsypkin's book mentioned in the footnote on the previous page. Now, the equation of the relay element (Figure 7.17a) is

$$x_{out} = \Phi(x)\, x_{in}, \tag{7.115}$$

where

$$\Phi(x) = \begin{cases} +x_{out} & \text{for} \quad x_{in} > 0, \\ -x_{out} & \text{for} \quad x_{in} < 0. \end{cases} \tag{7.116}$$

The characteristic (7.115) can be replaced by a continuous curve which has a finite derivative at the origin, $\Phi'(0) \neq \infty$ (Figure 7.19). The real characteristic is then obtained from that shown in Figure 7.19 by letting the angle β approach 90° and $\Phi'(0) \to \infty$. The variational equation of the relay is thus replaced by the equation of an inertialess amplifier of infinite gain:

$$x_{out} = \Phi'(0)\, x_{in}, \tag{7.117}$$

where

$$\Phi'(0) = K_c = \infty.$$

The structural diagram of the system corresponding to this variational equation is given in Figure 7.20, where the relay element has been replaced by an infinite-gain amplifier. The equation of the entire system can now readily be written.

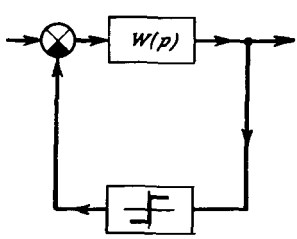

FIGURE 7.18. A block diagram of a system with a relay.

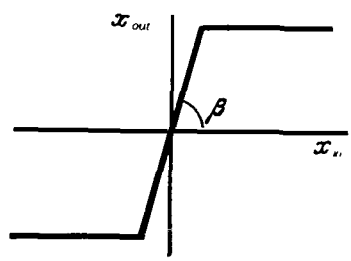

FIGURE 7.19. Illustrating the transition $K_c \to \infty$.

To avoid complications with stabilizers, we first consider a system with a single-loop linear part. The transfer function of the linear part is $\frac{K}{Q(p)}$.

The transfer function of the entire closed-loop system (see Figure 7.20) is then

$$\frac{\frac{KK_c}{Q(p)}}{1+\frac{KK_c}{Q(P)}} = \frac{KK_c}{Q(p)+KK_c}. \tag{7.118}$$

The characteristic equation is thus

$$\left.\begin{aligned} &Q(p)+KK_c=0 \\ \text{or} \quad &\frac{1}{K_c}Q(p)+K=0. \end{aligned}\right\} \tag{7.119}$$

It directly follows from (7.119) that the system is unstable if $Q(p)$ is of higher than second degree. When $Q(p)$ is of first or second degree, the system is stable if

$$\frac{B_1}{B_0}-\frac{A_1}{A_0}>0 \text{ or } \frac{B_0}{A_0}>0.$$

Both conditions are satisfied in our case provided that the coefficients of p and p^2 in $Q(p)$ are positive, since $A_1=0$.

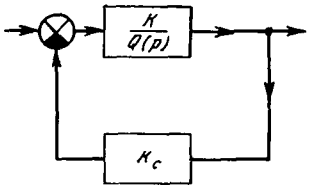

FIGURE 7.20. The equivalent circuit.

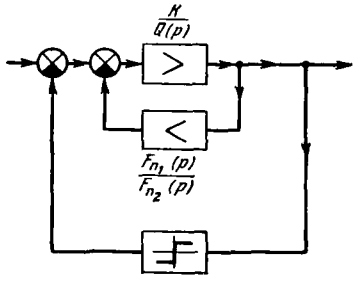

FIGURE 7.21. Illustrating the stability of a relay system.

Any additional finite time constant, however small, will render the system unstable. Therefore, in practice, configurations like the one in Figure 7.20 are unstable in the small.

Let now $Q(p)$ be of higher than second degree in p. A stabilizer is needed to make the system stable. It is clear, of course, that no stabilizer will do the job unless the relay element is also included in the stabilized loop. Indeed, Figure 7.21 is a structural diagram of a system where the stabilizer embraces only the linear part of the system. The transfer function of this structure is

$$I(p)=\frac{\frac{\frac{K}{Q(p)}K_c}{\frac{K}{Q(p)}\cdot\frac{F_{n_1}(p)}{F_{n_2}(p)}+1}}{1+\frac{\frac{K}{Q(p)}K_c}{\frac{K}{Q(p)}\cdot\frac{F_{n_1}(p)}{F_{n_2}(p)}+1}}. \tag{7.120}$$

The characteristic equation is obtained by putting the denominator of (7.120) equal to zero, thus:

$$Q(p)F_{n_2}(p)+KF_{n_1}(p)+K_cKF_{n_2}(p)=0. \tag{7.121}$$

Dividing both sides of (7.121) by K_c and putting $\frac{1}{K_c}=m$, we find

$$m[Q(p)F_{n_2}(p)+KF_{n_1}(p)]+KF_{n_2}(p)=0.$$

The difference in the degrees of the polynomial in brackets and the polynomial KF_{n_2} is determined, as before, by the degree of the polynomial $Q(p)$. This conclusion is not affected if only the linear part of the circuit is stabilized, the relay remaining outside the stabilizer loop. The only way to ensure stability is to let the stabilizer embrace the section that includes the relay element.

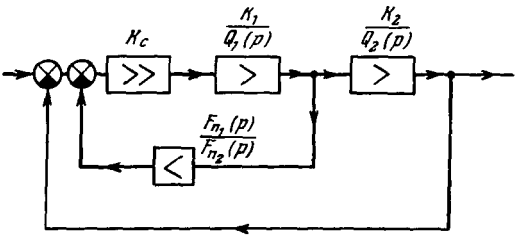

FIGURE 7.22. A more general case.

Figure 7.22 is a structural diagram of a control system where the stabilizer loop encloses the relay. In the nomenclature of Figure 7.22, we write the closed-loop transfer function in the form

$$W(p)=\frac{\dfrac{K_c\dfrac{K_1}{Q_1(p)}}{1+\dfrac{K_cK_1}{Q_1(p)}\cdot\dfrac{F_{n_1}(p)}{F_{n_2}(p)}}\cdot\dfrac{K_2}{Q_2(p)}}{1+\dfrac{K_c\dfrac{K_1}{Q_1(p)}\cdot\dfrac{K_2}{Q_2(p)}}{1+\dfrac{K_cK_1}{Q_1(p)}\cdot\dfrac{F_{n_1}(p)}{F_{n_2}(p)}}}. \tag{7.122}$$

The characteristic equation is thus

$$Q_1(p)Q_2(p)F_{n_2}(p)+K_c[K_1F_{n_1}(p)Q_2(p)+K_1K_2F_{n_2}(p)=0$$

or

$$mQ_1(p)Q_2(p)F_{n_2}(p)+[K_1F_{n_1}(p)Q_2(p)+K_1K_2F_{n_2}(p)]=0. \tag{7.123}$$

Let the polynomials $F_{n_2}(p)$, $F_{n_1}(p)$ and $Q_1(p)$be of the degree n_2, n_1, and q_1, respectively. The system is then stable for $m\to 0$ if

$$n_2+q_1-n_1\leqslant 2. \tag{7.124}$$

A system satisfying condition (7.124) is stable in the small if the degenerate equation

$$K_1 F_{n_1}(p)\, Q_2(p) + K_1 K_2 F_{n_2}(p) = 0$$

meets the stability requirements and the following conditions are fulfilled:

$$\frac{B_0}{A_0} > 0 \text{ if } n_2 + q_1 - n_1 = 1,$$

or

$$\frac{B_1}{B_0} - \frac{A_1}{A_0} > 0 \text{ if } n_2 + q_1 - n_1 = 2.$$

In order for a relay system to be stable in the small, it should be structurally stable at infinite gain. A relay is substituted in the limit for the infinite-gain amplifier.

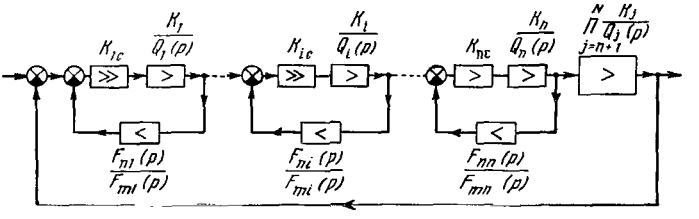

FIGURE 7.23. A system with n relays.

Let us consider a control system with n relay elements. This system can be made stable in the small with the aid of n stabilizers, which may be connected in two alternative configurations. Figure 7.23 is a structural diagram showing the stabilizers connected according to the first configuration. We will derive the transfer function of this system assuming small deviations. For the i-th relay and the linear section encompassed by the i-th stabilizer of a general kind $\frac{F_{ni}(p)}{F_{mi}(p)}$, the transfer function is written as

$$K(p) = \frac{\frac{K_i K_{ci}}{Q_i(p)}}{1 + \frac{K_i K_{ci}}{Q_i(p)} \cdot \frac{F_{ni}(p)}{F_{mi}(p)}} = \frac{K_i K_{ci} F_{mi}(p)}{Q_i(p) F_{mi}(p) + K_i K_{ci} F_{ni}(p)}. \tag{7.125}$$

Seeing that there is a total of n relay elements and putting $\prod_{j=n+1}^{N} \frac{K_j}{Q_j(p)}$ for the transfer function of the unstabilized section of the system, we obtain the following expression for the closed-loop transfer function:

$$K(p) = \frac{\prod_{i=1}^{n} Q_i(p) F_{mi}(p) + K_i K_{ci} F_{ni}(p) \prod_{j=n+1}^{N} \frac{K_j}{Q_j(p)}}{1 + \prod_{i=1}^{n} Q_i(p) F_{mi}(p) + K_i K_{ci} F_{ni}(p) \prod_{i=n+1}^{N} \frac{K_j}{Q_j(p)}} = \tag{7.126}$$

$$= \frac{\prod_{l=1}^{n} K_l K_{c\,l} F_{ml}(p) \prod_{j=n+1}^{N} K_l}{\prod_{l=1}^{n} [Q_l(p) F_{ml}(p) + K_l K_{c\,l} F_{nl}(p)] \prod_{j=n+1}^{N} Q_j(p) + \prod_{l=1}^{n} K_l K_{c\,l} F_{ml}(p) \prod_{j=n+1}^{N} K_j}. \quad (7.126)$$

Dividing the numerator and the denominator in (7.126) by K_c^n and putting $\frac{1}{K_c^{\nu}} = m^{\nu}$, we write

$$K(p) = \frac{\prod_{l=1}^{n} K_l \prod_{j=n+1}^{N} K_j \prod_{l=1}^{n} F_{ml}(p)}{m^n F_N(p) + m^{n-1} F_{N_1}(p) + m^{n-2} F_{N_2}(p) + \ldots + m F_{N_{n-1}}(p) + F_{N_n}(p)}, \quad (7.127)$$

where

$$F_N(p) = \prod_{l=1}^{n} Q_l(p) \prod_{j=n+1}^{N} Q_j(p) F_{ml}(p),$$

$$F_{N_1}(p) = \sum_{l=1}^{n} K_l F_{nl}(p) \ldots$$

$$\ldots \prod_{j=1,2,\ldots,l-1,l+1,\ldots,n} Q_j(p) F_{mj}(p) \prod_{j=n+1} Q_j(p),$$

. . . . . .

$$F_{N_n}(p) = \prod_{l=1}^{n} K_l F_{nl}(p) \prod_{j=n+1}^{N} Q_j(p) + \prod_{l=1}^{n} K_l F_{ml}(p) \prod_{j=n+1}^{N} K_j.$$

Putting the denominator of (7.127) equal to zero, we obtain the characteristic equation of the system. The stabilizers should be so chosen that the resulting configuration is structurally stable at infinite gain. Otherwise the system will be unstable. If the structure is correctly chosen, stability is actually ensured if the degenerate equation $F_{Nn}(p)$ and the auxiliary equations of first, second, or third kind each satisfy the corresponding stability conditions.

If the gain remains high in the case of large deviations too, the transient behavior of the system is determined by the transfer function obtained from (7.127) putting $m = 0$, i.e.,

$$K(p) = \frac{\prod_{l=1}^{n} K_l \prod_{j=n+1}^{N} K_j \prod_{l=1}^{n} F_{ml}(p)}{\prod_{l=1}^{n} K_l F_{nl}(p) \prod_{j=n+1}^{N} Q_j(p) + \prod_{l=1}^{n} K_l F_{ml}(p) \prod_{j=n+1}^{N} K_j} =$$

$$= \frac{\prod_{j=n+1}^{N} K_j}{\prod_{l=1}^{n} \frac{F_{nl}(p)}{F_{ml}(p)} \prod_{j=n+1}^{N} Q_j(p) + \prod_{j=n+1}^{N} K_j}. \quad (7.128)$$

In fact we can speak only of some averaged gain, which is determined by the ratio of the relay output to input. The higher the input, the lower is the averaged gain. Near the origin, even the averaged gain is fairly high.

Let us now concentrate on the physics of the process in an n-relay system with structural infinite-gain stability. There are two possibilities:

(a) The stabilizer is such that each small parameter $m=\frac{1}{K_c}$ raises the degree of the characteristic equation by one. In particular, if the stabilizer ratio is

$$I(p)=\frac{\tau p}{1+\tau p}$$

and the relay is inertialess, the forward path of the loop formed by the relay and the stabilizer may include only a single element with a first-order equation.

In the general case, the transfer function of a closed loop comprising a linear element $\frac{K_l}{Q_l(p)}$, a relay with a gain K_{cl}, and a stabilizer $\frac{F_{nl}(p)}{F_{ml}(p)}$ is written in the form

$$K(p)=\frac{\frac{K_l}{Q_l(p)}K_{cl}}{1+\frac{K_l}{Q_l(p)}\cdot K_c\frac{F_{nl}(p)}{F_{ml}(p)}}=\frac{K_lK_{cl}F_{ml}(p)}{Q_l(p)F_{ml}(p)+K_lK_{cl}F_{nl}(p)}. \qquad (7.129)$$

If there are n such loops, and the total number of elements is N, the transfer function is

$$K(p)=\frac{\prod_{l=1}^{n}\frac{K_lK_{cl}F_{ml}(p)}{Q_l(p)F_{ml}(p)+K_lK_{cl}F_{nl}(p)}\prod_{j=l+1}^{N}\frac{K_j}{Q_j(p)}}{1+\prod_{l=1}^{n}\frac{K_lK_{cl}F_{ml}(p)}{Q_l(p)F_{ml}(p)+K_lK_{cl}F_{nl}(p)}\prod_{j=n+1}^{N}\frac{K_j}{Q_j(p)}}. \qquad (7.130)$$

The characteristic equation is obtained by putting the denominator of (7.130) equal to zero:

$$\prod_{l=1}^{n}[Q_l(p)F_{ml}(p)+K_lK_{cl}F_{nl}(p)]\prod_{j=n+1}^{N}Q(p)+\prod_{l=1}^{n}K_lK_{cl}F_{ml}(p)\prod_{j=n+1}^{N}K_j=0. \qquad (7.131)$$

Dividing (7.131) through by $\prod_{l=1}^{n}K_lK_{cl}$ and putting $\frac{1}{K_lK_{cl}}=m_l$, we assume that K_l and $K_1=K$ are related by $K_l=\eta_lK_1=\eta_lK$ and thus write (7.131) in the form

$$m^nF_{N_1}(p)+m^{n-1}F_{N_0-1}(p)+m^{n-2}F_{N_0-2}+\ldots$$
$$\ldots+mF_{N-n+1}(p)+F_{N_0-n}(p)=0. \qquad (7.132)$$

Here N_0 is the degree of the characteristic equation, equal to $\sum_{l=1}^{N}q_l+\sum_{l=1}^{n}n_{2l}$, and the subscripts of F designate the degree of the corresponding polynomial. The transfer function (7.130) for this case takes the form

$$K_1(p)=\frac{\prod_{j=n+1}^{N}K_j\prod_{l=1}^{n}F_{ml}(p)}{m^nF_{N_1}(p)+m^{n-1}F_{N_0-1}(p)+\ldots+mF_{N_0-n+1}(p)+F_{N-n}(p)}. \qquad (7.133)$$

(b) The stabilizer is such that each small parameter $m=\frac{1}{K_1K_c}$ increases the degree of the equation by two. For the particular case of a stabilizer with a transfer function $\frac{\tau p}{1+\tau p}$ and an inertialess relay, the forward path of the loop formed by the relay and the stabilizer may include a single second-order element (e.g., an oscillator) or two first-order elements (aperiodic or integrating).

In the general case, making use of the notation in (a), we write the characteristic equation in the form

$$m^nF_N(p)+m^{n-1}F_{N_0-2}(p)+m^{n-2}F_{N-4}(p)+\ldots+F_{N-2n}(p)=0. \qquad (7.132)'$$

The corresponding transfer function is

$$K_2(p)=\frac{\prod_{j=2n+1}^{N}K_j\prod_{i=1}^{n}F_{mi}(p)}{m^nF_{N_0}(p)+m^{n-1}F_{N_0-2}(p)+m^{n-2}F_{N_0-4}(p)+\ldots+F_{N_0-2n}(p)}. \qquad (7.133)'$$

A relay with the unit step characteristic shown in Figure 7.17a is stable only if the input signal is zero ($x=0$). Integrating systems are therefore assumed in both cases, which are in equilibrium for $x=0$.

2. Sliding mode

Sliding-action relay systems have been investigated in considerable detail /11, 17, 13/. The physics of the sliding mode has been established and the relationships to be satisfied for a system to operate in the sliding mode have been derived.

We are interested in sliding action in connection with the following problem. The use of structures with infinite-gain stability always raises the question of how the infinite gain is to be realized. In the great majority of cases the gain values for which all the preceding results hold true are readily attainable, as they fall in the range of common gain values of control systems (300—1000).

We have shown for saturable nonlinearities with a positive slope factor in the saturation zone that introduction of an amplifier of sufficiently high gain in series with the nonlinear element transforms the system to a high-gain structure. In the case of a relay characteristic, the slope in the saturation region is zero. It has been shown in the literature /71/ and will be demonstrated in the following that a sliding-action relay system is equivalent to an infinite-gain linear system.

We now proceed to synthesize a structure which will be equivalent to a linearized system and derive the equations that describe its dynamics. Figure 7.24 is a block diagram of a relay system.

Nomenclature:

$K(p)$= the transfer function of the unstabilized section;
$F_s(p)$= the stabilizer transfer function;
R.E.= a relay element with an ideal characteristic (without an insensitive zone);

$K_1(p)$ = the transfer function of the stabilized element in series with RE;
$Y_{ref}(p)$ = the transform of the reference signal;
$Y(p)$ = the transform of the output;
$X(p) = Y_{ref}(p) - Y(p)$;
$Z(p)$ = the transform of the stabilizer output.

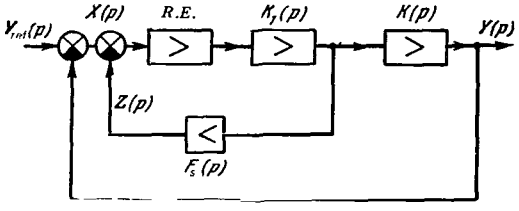

FIGURE 7.24. Illustrating the sliding-action conditions.

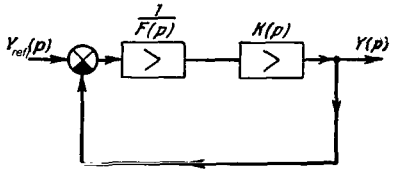

FIGURE 7.25. A linearized system.

Let the system operate in the sliding mode. Then, as was shown in /8, 71/, the relay element oscillates at infinite frequency with infinitesimal amplitude. The relay input $x(t)$ can be taken as zero, so that

$$x(t) = y_{ref}(t) - y(t) - z(t) = 0, \tag{7.134}$$

and this is equivalent to an infinite-gain relay. The linear equivalent of this relay system is thus a structure where an infinite-gain linear amplifier is substituted for the relay element. The degenerate part of this linear system is equivalent to a sliding-action relay system.

Structurally, the linearized system can be depicted as in Figure 7.25. We see from the diagram that the input signal in this case is

$$x_1 = y_{ref}(t) - y(t). \tag{7.135}$$

The equation for $x_1(t)$ in Laplace transforms is

$$\mathcal{L}\{x_1(t)\} = \frac{1}{1 + \frac{K(p)}{F_s(p)}} \mathcal{L}\{y_{ref}(t)\}$$

or

$$\mathcal{L}\{x_1(t)\} = \frac{F_s(p)}{F_s(p) + K(p)} \mathcal{L}\{y_{ref}(t)\}, \tag{7.136}$$

and the equation for the output $y(t)$ is obtained by inserting for $x_1(t)$ in (7.136) its expression from (7.135), thus·

$$\mathcal{L}\{y(t)\} = \frac{K(p)}{F_s(p) + K(p)} \mathcal{L}\{y_{ref}(t)\} \ldots \tag{7.137}$$

The preceding considerations are meaningful if the system remains stable, i.e., if the conditions of infinite-gain stability are fulfilled.

Our analysis shows that continuous sliding action is possible if at any time the external impulse $x_i(t)$ varies at a slower rate than the internal-feedback impulse $z(t)$, i.e., if

$$|\dot{x}_i(t)| < |\dot{z}(t)|, \tag{7.138}$$

and it is only on this condition that the relay system can be replaced by a linear equivalent.

Following Ya.Z. Tsypkin, we proceed to determine the condition of existence of continuous sliding action in terms of system parameters and external impulses. We first have to express $z(t)$ and $x_1(t)$ in explicit form. From Figure 7.24 we see that

$$\mathcal{L}\{z(t)\} = K_1(p)\,F_s(p)\,\mathcal{L}\{\Phi(0)\}, \tag{7.139}$$

but since

$$\mathcal{L}\{\Phi(0)\} = \mathcal{L}[\pm K_c] = \pm\frac{K_c}{p},$$

we have

$$\mathcal{L}\{z(t)\} = \pm K_1(p)\,F_s(p)\,\frac{K_c}{p}. \tag{7.140}$$

We see from (7.140) that $z(t)$ depends on the parameters of the internal loop. As regards $x_1(t)$, we have from (7.136)

$$\mathcal{L}\{x_1(t)\} = \frac{F_s(p)}{F_s(p)+K(p)}\,\mathcal{L}\{y_{ref}(t)\}.$$

Making use of the known properties of the Laplace transform, we write

$$\left.\begin{aligned}\mathcal{L}\{\dot{z}(t)\} &= p\mathcal{L}\{z(t)\} - z(0),\\ \mathcal{L}\{\dot{x}(t)\} &= p\mathcal{L}\{x(t)\} - x_1(0).\end{aligned}\right\} \tag{7.141}$$

Inserting for $\mathcal{L}\{z(t)\}$ and $\mathcal{L}\{x(t)\}$ their expressions from (7.139) and (7.136), we find

$$\mathcal{L}\{\dot{z}(t)\} = \pm\frac{K_c}{p}K_1(p)\,F_s(p) - z(0), \tag{7.142}$$

$$\mathcal{L}\{\dot{x}(t)\} = \frac{pF_s(p)}{F_s(p)+K(p)}\,\mathcal{L}[y_{ref}(t)] - z(0). \tag{7.143}$$

If $z(t)$ and $\dot{x}(t)$ are determined from (7.142), (7.143) and the results are substituted in (7.138), we obtain the conditions of continuous sliding action.

A sliding-action relay system is thus equivalent to an infinite-gain linear system. A relay system in the sliding mode can thus be regarded as an example of a real system with arbitrarily high gain. This approach is often very convenient for systems where amplifiers have relay characteristics.

§ 7.12. THE PROBLEM OF SENSITIVITY

One of the methods to synthesize fixed-structure systems equivalent to adaptive systems is by choosing a configuration where the principal dynamic properties are independent of a wide-range variation of certain plant parameters or even of certain characteristics of system components.

Bode was the first to introduce the concept of sensitivity, which essentially determines to what extent a change in the parameters of the individual elements affects the dynamics of the system as a whole. This approach has established an intimate relationship between the synthesis of fixed-structure systems equivalent to adaptive systems and the design of structures which are insensitive or little sensitive to variation between wide limits of plant parameters, plant characteristics, or characteristics of individual system elements.

The problems treated in this book are directly related to the various topics which are considered in the literature* under the separate heading of control system design. In this category, e.g., there is the problem of a low-damping oscillatory plant, of the so-called zero-sensitivity systems**, where positive and negative feedback are used simultaneously, etc. As regards the achievement of zero sensitivity by simultaneous application of positive and negative feedback, it is shown in Chapter Six that, unless special measures are taken, this solution yields noncoarse systems (in the sense of A.A. Andronov). However, the main point here is that the synthesis of systems which are insensitive to variation of parameters and characteristics of the controlled object or of some component elements is an inherently structural problem. A feedback system is not only an illustration but a convincing proof that the desired properties are ensured only by appropriately designed structures. We know that the sensitivity to parameter variation in a negative feedback loop diminishes as the gain is increased. However, increase of gain may lead to system instability. The problem is therefore again to synthesize a structure which will ensure the necessary gain without losing its overall stability.

Absolute or relative changes in the dynamic properties of the system as a function of parameter increments can be used as sensitivity indices. Bode introduced the following sensitivity index with a definite physical meaning. Let $K(p)$ be the closed-loop transfer function. The sensitivity is defined as the ratio of the change in the closed-loop transfer function to the change in the plant transfer function, i.e.,

$$S_{W_2(p)}^{K(p)} = \frac{d \ln K(p)}{d \ln W_2(p)} = \frac{dK(p)}{dW_2(p)} \cdot \frac{W_2(p)}{K(p)}. \tag{7.144}$$

An alternative definition of the sensitivity coefficient has been advanced by P. Kokotović /86/:

$$S = \frac{dK(p)}{d \ln q} = \frac{dK(p)}{dq} q, \tag{7.145}$$

* Bode, H.W. Network Analysis and Feedback Amplifier Design. — N.Y., Van Nostrand. 1945.

** Truxal, J.G.—In: "Samoprisposablivayushchie sistemy", Ch. III. Translations from English. Moskva IL. 1963.

where q is the parameter whose influence on system dynamics is being considered.

In what follows we will describe the application of the two definitions to particular cases.

1. Bode sensitivity S_W^K in single-variable systems

Plant characteristics can be altered only by external disturbances. The characteristics change in the result of on-line interference from some of the plant parameters. We are concerned here with systems where the plant parameters can vary between fairly wide limits in the course of the control process. An adaptation (or self-adjustment) index of these systems is the degree of insensitivity of the transients to the plant properties or, more precisely, to their variation.

We will apply the Bode sensitivity S_W^K to estimate the adaptivity of systems with the above properties. In fact, the smaller the sensitivity S_W^K, the closer is the system to an ideally adaptive one. At first we consider a synthesis technique utilizing minimum-S_W^K structures for single-variable control systems.

As we have already observed, the properties of a system become progressively insensitive to changes in the controlled object as $S_{W_2(p)}^{K(p)}$ decreases. For this reason an ideal adaptive structure is such that $S_{W_2(p)}^{K(p)}$ is independent of $W_2(p)$ or $S_{W_2(p)}^{K(p)} \to 0$.

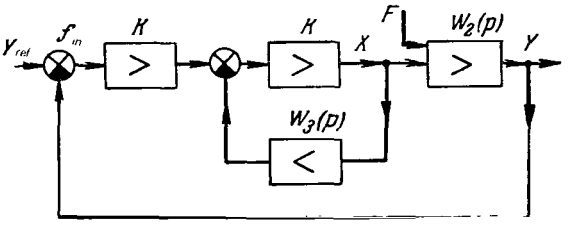

FIGURE 7.26. An adaptive system.

We now prove the following proposition: structures with infinite-gain stability stabilized by nearly ideal derivatives, where the derivatives are "idealized" by adjusting the gain values, are adaptive in the above sense. Indeed, consider a structure of this kind, shown in Figure 7.26. The closed-loop transfer function is

$$K(p) = \frac{\frac{K}{W_3(p)} W_2(p)}{1 + \frac{K}{W_3(p)} W_2(p)}. \qquad (7.146)$$

The sensitivity (7.144) is thus given by

$$S^{K(p)}_{W_2(p)} = \frac{\frac{K}{W_3(p)}\left[1+\frac{K}{W_3(p)}W_2(p)\right]-\frac{K}{W_3(p)}-\frac{K}{W_3(p)}W_2(p)}{\left[1+\frac{K}{W_3(p)}W_2(p)\right]^2} \times \frac{1+\frac{K}{W_3(p)}W_2(p)}{\frac{K}{W_3(p)}}. \quad (7.147)$$

Simplifying, we find

$$S^{K(p)}_{W_2(p)} = \frac{1}{1+\frac{K}{W_3(p)}W_2(p)} \quad (7.148)$$

and

$$\lim_{K\to\infty} S^{K(p)}_{W_2(p)} = 0.$$

We have obtained an ideal system in the sense of the preceding.

Now consider the expression for sensitivity of systems with infinite-gain stability stabilized by passive stabilizers.

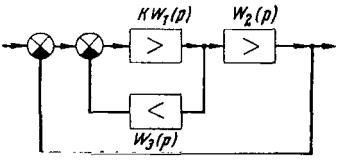

FIGURE 7.27. Structural equivalent of Figure 7.26.

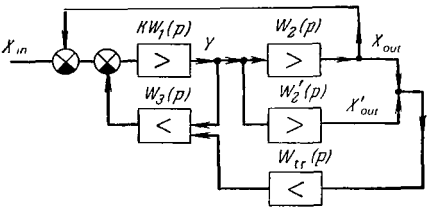

FIGURE 7.28. The use of an ideal plant.

As an example we take the simple case of a system shown in Figure 7.27. The closed-loop transfer function is

$$K(p) = \frac{KW_1(p)\,W_2(p)}{1+KW_1(p)\,W_3(p)+KW_1(p)\,W_2(p)}. \quad (7.149)$$

The sensitivity is given by

$$S^{K(p)}_{W_2(p)} = \frac{[1+KW_1(p)\,W_3(p)+KW_1(p)\,W_2(p)]-KW_1(p)\,W_2(p)}{1+KW_1(p)\,W_3(p)+KW_1(p)\,W_2(p)}. \quad (7.150)$$

Simplifying, we find

$$S^{K(p)}_{W_2(p)} = \frac{1+KW_1(p)\,W_3(p)}{1+KW_1(p)\,W_3(p)+KW_1(p)\,W_2(p)}.$$

At sufficiently high gain, we have in the limit

$$\lim_{K\to\infty} S^{K(p)}_{W_2(p)} = \frac{W_3(p)}{W_2(p)+W_3(p)}. \quad (7.151)$$

We see from (7.151) that even at fairly high gain, the system dynamics remains sensitive to changes in plant parameters or characteristics.

Let us now try to improve on the stable structure so as to minimize the parameter influence on system dynamics and to make the system ideally adaptive in the above sense.

As in the case of external disturbances which defy measurement, the adaptive system can be conveniently synthesized with the aid of an ideal, noisefree plant model. Figure 7.28 is a structural diagram corresponding to this case. Using the nomenclature of Figure 7.28, we write

$$I(p) = KW_1(p)\{X_{in}(p) - X_{out}(p) - W_3(p)Y(p) - \\ - W_n(p)W_3(p)[X'_{out}(p) - X_{out}(p)]\}. \quad (7.152)$$

Here $X'_{out}(p)$ is the transform of the ideal output, $X_{out}(p)$ the transform of the real plant output. The ideal plant characteristics are assumed to remain constant.

The difference $X'_{out}(p) - X_{out}(p)$ is thus equivalent to a disturbance due to variation of plant characteristics. Hence,

$$X'_{out}(p) - X'_{out}(p) = cF(p), \quad (7.153)$$

where c is a constant. Thus,

$$X_{out}(p) = W'_2(p)Y(p) + cW_2(p)F(p). \quad (7.154)$$

Inserting for $X'_{out}(p) - X_{out}(p)$ in (7.152) its expression from (7.153), we find

$$Y(p) = KW_1(p)[X_{in}(p) - X_{out}(p)W_3(p)Y(p) - W_{np}(p)W_3(p)cF(p)], \quad (7.155)$$

whence

$$Y(p) = \frac{KW_1(p)X_{in}(p) - KW_1(p)X_{out}(p) - KW_1(p)W_{np}(p)W_3(p)cF(p)}{1 + KW_1(p)W_3(p)}. \quad (7.156)$$

Substituting (7.156) in (7.154), we find

$$X_{out} = \frac{W'_2(p)[KW_1(p)X_{in}(p) - KW_1(p)X_{out}(p) - W_{np}(p)W_3(p)KW_1(p)cF(p)]}{1 + KW_1(p)W_3(p)} \quad (7.157)$$

and solving for $X_{out}(p)$ we write

$$X_{out}(p) = \frac{KW_1(p)W'_2(p)X_{in} + W'_2(p)cF(p)}{1 + KW_1(p)W'_2(p) + KW_1(p)W_3(p)}. \quad (7.158)$$

For $K \to \infty$ we have

$$X_{out}(p) = \frac{W_1(p)W'_2(p)X_{in}(p)}{W_1(p)W_3(p) + W_1(p)W'_2(p)} = \frac{W'_2(p)X_{in}(p)}{W_3(p) + W'_2(p)}. \quad (7.159)$$

We see from (7.159) that the output is insensitive to changes in plant parameters. If $W'_2(p)$ is optimized (with respect to some quality criterion), the system will hold the optimum irrespective of changes in plant characteristics.

2. Kokotović sensitivity S

We deal here with the same cases as in the preceding subsection, using the sensitivity (7.145). The variable parameter is the plant gain, which is allowed to drift between wide limits. The transfer function of the plant is

$$W_2(p) = K_0 W_2'(p); \tag{7.160}$$

relation (7.146) is thus written as

$$K(p) = \frac{\frac{KK_0 W_2'(p)}{W_3(p)}}{1 + \frac{KK_0 W_2'(p)}{W_3(p)}}. \tag{7.161}$$

We are concerned with sensitivity in respect to relative changes in plant gain, K_0. We have

$$S = \frac{\frac{KW_2'(p)}{W_3(p)}\left[1 + \frac{KK_0 W_2'(p)}{W_3(p)}\right] - \frac{KW_2'(p)}{W_3(p)} \cdot \frac{KK_0 W_2'}{W_3(p)}}{\left[1 + \frac{KK_0 W_2'(p)}{W_3(p)}\right]^2} K_0 = \frac{KW_2'(p) W_3(p)}{\left[W_3(p) + KK_0 W_2'(p)\right]^2}. \tag{7.162}$$

Thus,

$$\lim_{K \to \infty} S = 0, \tag{7.163}$$

i.e., the same result as before. If the system is stabilized by passive stabilizers (real derivatives), the results are also the same as those obtained with the Bode sensitivity. In this class of structures, S_W^K and S are equivalent in the sense that they give identical results.

The principal structural conclusion that follows from the preceding can be formulated as follows. In order for the system dynamics to be independent of changes in parameters or characteristics of some element, the controlled object included, it is necessary that the gain of the loop with the variable element be sufficiently high. It is implied that the entire system remains stable in the process. A system of this kind is realizable if its structure possesses infinite-gain stability.

We see from our preceding treatment of sensitivity in two structures with infinite-gain stability that, in the second case, increase of gain failed to produce sufficiently low sensitivity without the incorporation of an ideal noisefree plant. This was so because we did not increase the gain of the loop with the variable element.

In practice, low-sensitivity systems can be synthesized by a simultaneous application of the two techniques. This will enable us to dispense with the ideal plant in the network. As an illustration, let us consider the case of a structure which is stabilized by ordinary passive elements. The structure in Figure 7.26 is modified as follows (Figure 7.29): a high-gain amplifier is connected in series with the variable-parameter plant. The closed-loop transfer function for Figure 7.29 is

$$K(p) = \frac{K^2 W_1(p) W_2(p)}{1 + KW_1(p) W_3(p) + K^2 W_2(p) W_1(p)} \tag{7.164}$$

and

$$\lim_{K\to\infty} S^{K(p)}_{W_2(p)} = 0, \tag{7.165}$$

i.e., we have obtained a system with zero sensitivity.

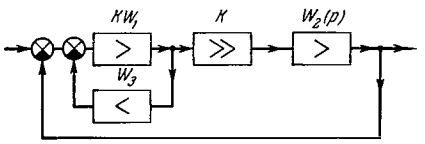

FIGURE 7.29. An adaptive system.

It remains to be shown, however, that the system is stable as $K\to\infty$. Let the transfer functions of the system elements be

$$KW_1(p) = \frac{R_1(p)}{Q(p)}, \qquad W_3(p) = \frac{F_n(p)}{F_m(p)} \quad \text{and} \quad W_2(p) = \frac{K_0}{D(p)}.$$

The characteristic equation is obtained by putting the determinant of (7.164) equal to zero, thus:

$$1 + K\frac{R(p)}{Q(p)} \cdot \frac{F_n(p)}{F_m(p)} + K^2 \frac{K_0}{D(p)} \cdot \frac{R(p)}{Q(p)} = 0. \tag{7.166}$$

Dividing (7.166) through by K^2 and putting $\frac{1}{K} = m$, we obtain after simple manipulations

$$m^2 D(p) Q(p) F_m(p) + mR(p) F_m(p) D(p) + K_0 R(p) F_m(p) = 0. \tag{7.167}$$

The difference in the degrees of the first two polynomials is $q+m-r-n$, where q, m, r, and n are the respective degrees of the polynomials Q, F_m, R, and F_n. Since the structure in Figure 7.28 has infinite-gain stability, we may write

$$q + m - r - n \leqslant 2.$$

Now consider the difference ν in the degrees of the last two polynomials. If d is the degree of $D(p)$, we have

$$\nu = n + d - m.$$

Since $n - m$ has been determined from the structure in Figure 7.28, everything depends on the value of d, which is the degree of the denominator of the plant operator. If $d \leqslant 2$, zero sensitivity can be attained for the structure in Figure 7.29 without any additional means. If, however, $d>2$, the inequality

$$\nu = n - m + d \leqslant 2$$

must be satisfied. This can be done by a simultaneous application of the first and second methods of synthesis of previously discussed structures which are stable for $K \to \infty$.

We have already shown how to achieve infinite-gain stability in systems with nonlinearities of a certain kind. We have also emphasized that an infinite gain is realizable with a sliding-action relay system. Infinite gain can also be obtained with the aid of a sliding-action system of variable structure /8/.

At the end of this chapter we will show by considering a number of examples that our conclusions concerning zero-sensitivity structures can be extended to plants with variable parameters as well.

And now a few words on multivariable control systems. In this section a system is regarded as ideal or adaptive if the control dynamics are not overly influenced by the variation of plant characteristics. Feedback between the controlled variables obviously affects the dynamics in each control loop irrespective of whether the particular controlled variable is sensitive to variation of plant characteristics in the other variables or not. For this reason, system optimization in this case automatically involves noninteraction. If each controlled variable has its own extremum, and there is no single extremum for the entire system, noninteraction is the most desirable operating mode.

§ 7.13. SYSTEMS CONTAINING ELEMENTS WITH VARIABLE PARAMETERS

The parameters of many elements vary with time. Systems containing such variable elements are called systems with variable parameters. The time variation of the parameters may be quite arbitrary. For example, the self-inductance and the mutual inductance of synchronous machines with prominent poles are sine functions. In general, time variation of the parameter is not always known. If an element with a variable parameter is included in a control system, the variation can be interpreted as internal parametric noise, an obviously undesirable effect. We thus again arrive at a problem of sensitivity: find a structure such that time variation of a parameter does not influence the dynamic properties of the system as a whole or, alternatively, find a structure whose dynamic properties are insensitive to time variation of the parameters of individual elements.

Consider the following example. Let the controlled object be described by a first-order linear differential equation with variable parameters, specifically:

$$a(t)\frac{dy_{out}}{dt} + b(t)y_{out} = x. \qquad (7.168)$$

Here $a(t)$ and $b(t)$ are time-variable coefficients, y_{out} a controlled variable, x the controller input. Our task is to maintain y_{out} constant.

We will make use of the previous results obtained for linear systems with constant parameters. Figure 7.30 is a structural diagram of a system

that maintains the output y_{out} constant, changing it only if y_{ref} is changed. We put $\frac{d}{dt}=p$.* Then (7.168) is written

$$[a(t)p+b(t)]y_{out}(t)=x(t). \tag{7.169}$$

The equation of the system shown in Figure 7.30 is written as follows. For the section with constant coefficients we have

$$\frac{KK(1+Tp)}{Tp+1+K}[y_{ref}(t)-y_{out}(t)]=x(t), \tag{7.170}$$

and the element with variable parameters is described by (7.169). Eliminating $x(t)$ and differentiating, we find

$$\begin{aligned}[Ta(t)p^2+Tpa(t)p+Tb(t)p+Tpb(t)+a(t)p+b(t)]y_{out}(t)+\\ +K[a(t)p+b(t)]y_{out}(t)+K^2(1+Tp)y_{out}(t)=\\ =K^2(1+Tp)y_{ref}(t).\end{aligned} \tag{7.171}$$

Dividing both sides of (7.171) by K^2 and putting $\frac{1}{K}=m$, we find

$$\left.\begin{aligned}&\{m^2[Ta(t)p^2+(Tpa(t)+a(t)+Tb(t))p+\\ &\quad +Tpb(t)+b(t)]+m[a(t)p+b(t)]+\\ &\quad +(1+Tp)\}y_{out}(t)=(1+Tp)y_{ref}(t),\\ &\lim_{m\to 0}y_{out}(t)=y_{ref}(t).\end{aligned}\right\} \tag{7.172}$$

We have automatically obtained an ideal response, provided that the gain can be made arbitrarily large. The only restriction imposed in this case is the requirement of continuity of the time-dependent coefficients. The absolute values of $pb(t)$ and $pa(t)$ are thus finite. This restriction on the variation of the coefficients enables us to introduce further simplifications and to elucidate in greater detail the dynamic properties of the system.

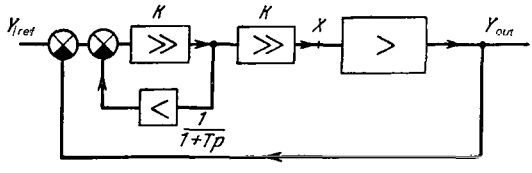

FIGURE 7.30. A plant with variable parameters.

Indeed, introduction of high-gain amplifiers, one of which is embraced by an aperiodic element, ensures faster and more faithful reproduction of the reference signal as the gain is increased, so that for sufficiently high gain values the coefficients $a(t)$ and $b(t)$ can be regarded as slowly varying. In a sense we end up with a network which is equivalent to a

* Do not confuse this operator p with the complex number in Laplace transformation.

linear structure with constant coefficients. The structure chosen should remain stable at infinite gain. The degenerate equation then takes the form

$$[T+m\overline{a(t)}]\,p+1+\overline{b(t)}\,m=0, \tag{7.173}$$

where $\overline{a(t)}$ and $\overline{b(t)}$ are constant for the duration of the transient. If $\overline{a(t)}$ and $\overline{b(t)}$ may take on negative values, it is necessary for the stability of the degenerate equation that $m\overline{a(t)}<T$ and $m\overline{b(t)}<1$, which is always feasible by making an appropriate gain adjustment.

The additional condition in this case has the form

$$\frac{T\overline{a}(t)}{T+m\overline{a}(t)}>0. \tag{7.174}$$

We see from (7.174) that the coefficient $a(t)$ must not be negative; otherwise the system is unstable.

We have thus proved that a virtually ideal response is attainable in this class of structures in the presence of elements with time-variable parameters. In other words, we have obtained a structure which is insensitive to the influence of time-variable parameters.

The above results can be readily generalized to the case of a controlled object described by an n-th order equation with variable parameters.

If n is the order of the equation describing the variable element, there is in general $n+2$ variable parameters, and a dynamically insensitive structure is generated by connecting n amplifiers of sufficiently high gain in series with the variable elements. Of these, $n-1$ amplifiers are stabilized by feedback elements of the type $\frac{1}{1+Tp}$. The system is tested for stability assuming relatively slow variation of the coefficients.

The system is realizable if the degenerate and the auxiliary equation each satisfy the stability conditions. The number of amplifiers may be reduced to $n/2+1$ if each amplifier is stabilized by a device with a transfer function $\frac{1}{ap^2+bp+1}$. This produces an auxiliary equation of the third kind. In practice, it is more advisable to use n amplifiers for the following reasons. First, the amplifiers themselves have a certain, albeit small, inertia, and this may limit the gain if $\frac{1}{ap^2+bp+1}$ stabilizers are used, while $\frac{1}{Tp+1}$ amplifiers are virtually unaffected by this property. As an illustration, Figure 7.31 gives a specimen structure for the case of a plant described by a fourth-order differential equation with variable parameters.*

One highly important property of these structures should be stressed. The point is, that the effect of the variable parameters on system dynamics is suppressed by the gain of the unstabilized amplifier. As regards the other amplifiers, they produce the derivative action required for purposes of stabilization (it also ensures accurate and fast response). If there are n amplifiers with stabilizers of the type $\frac{1}{Tp+1}$, all derivatives from n-th

* If the time rate of parameter variation cannot be ignored, the stability should be investigated by the method of V. M. Popov.

to first are produced; the higher the gain values, the closer are these derivatives to the ideal. But the gain of a closed loop comprising a high-gain amplifier and a stabilizer is close to unity. This is highly significant for noisy systems.

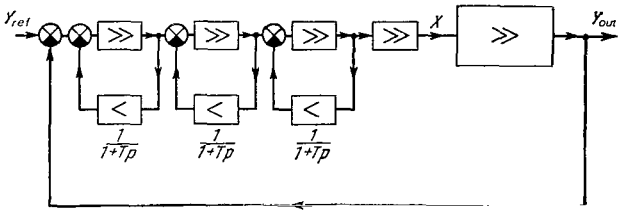

FIGURE 7.31. A more general case of a plant with variable parameters.

High gain is attained with the aid of sliding-action relay systems /71/ or variable-structure systems, also operating in the sliding mode /8/.

In conclusion a few words on the potential of the systems discussed.

From the aspect of classification of optimum control systems (e.g., according to Draper and Lee), we have to consider two cases.

1. The plant characteristics and the input-output functional dependence are well known. One input is adopted as the primary reference for control purposes, and all other inputs are generated by a programmed device which optimizes the system in accordance with the given input-output relationships. This system will function successfully in the noisefree case or if noise is suppressable.

2. The plant characteristic is not known. We only know that it has an extremum, which can be located by one of the searching techniques. First, the characteristics of the searching signal should be optimized in terms of gain and frequency; second, the output searching losses are minimized (this is the difference between the optimum value and the effective steady-state output); third, the time-to-optimum is minimized, and last, the realizability of a system which only requires occasional search is established.

Let us consider the case of a plant characteristic represented by the curve in Figure 7.32. For small deviations from the extremum, the characteristic is satisfactorily approximated by a parabola

$$y = Kx^2. \tag{7.175}$$

This particular assumption does not detract from the generality of our conclusions. It should be stressed, however, that the assumption expressed by equation (7.175) is physically meaningful. It implies that the structures we are interested in are potentially capable of ensuring very high transient and steady-state accuracy. Assumption (7.175) is thus fully justified.

In § 7.5 we have assumed that the noises altering the plant characteristics are injected directly into the plant and that they can be measured.

In this case neither continuous nor periodic search is required. It suffices to find once and for all the optimizing parameters, and the system is then synthesized as a combined control system along the lines described in Chapter Five. In reality, however, even if the relevant noise is delivered to the plant input, we cannot be sure that some other disturbance will not cause the output to drift from the optimum; the probability of this drift is the same whether the input is decreased or increased. Making use of (7.175), we readily see that drift due to a decrease of the input substantially alters the properties of the entire system, since the plant characteristic is unstable under these conditions and all the calculations should be carried out keeping this instability in mind.

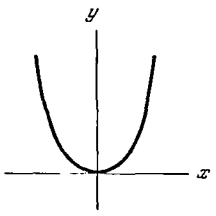

FIGURE 7.32. A plant characteristic.

For the system to retain the same structure in all operating modes, a combined-action system should be built, where the controllable deviations are no longer the deviations of the output from the reference value but the deviations of $\frac{dy_{out}}{dy_{in}}$ from zero.

It is significant, however, that the proposed fixed-structure systems are essentially different from ordinary searching systems in the following particulars.

1. Since the main noise is suppressed, the search characteristics are chosen so that the searching region and the output searching loss are minimized.
2. Periodic search is quite sufficient· it is turned on only when the controlled variable has departed from the optimum by more than a preset permissible value.
3. A successful synthesis technique calls for a combination of extremum-holding systems with periodic search.

§7.14. SPECIMEN CALCULATION OF A FIXED-STRUCTURE CONTROL SYSTEM WITH SELF-ADAPTIVE PROPERTIES

The example discussed in this section is borrowed from R.J. Kochenburger's paper presented at the IFAC Second Congress.

Figure 7.33 is a block diagram of Kochenburger's system (the symbols have been altered to conform with the usage in this book. The problem is to maintain the controlled variable y constant and equal to the reference value y_{ref}. The transfer function of the controlled object is $\frac{K}{D(p)}$; the parameters of the operator $D(p)$ remain strictly constant, while the gain K varies between wide limits. In Kochenburger's system the gain varies by a factor of 100 : 1, and it is this gain variation that provides the main disturbance.

The author rightly stresses that his solution is considerably simpler than the conventional solutions, where complex calculators are used to perform the search. In Kochenburger's solution the product μK is maintained constant (μ is the controller gain). Therefore, there is an element measuring the change in K an another element which alters μ appropriately, so that $\mu K = \text{const}$.

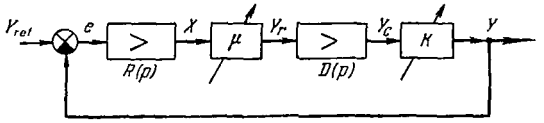

FIGURE 7.33. Kochenburger's system.

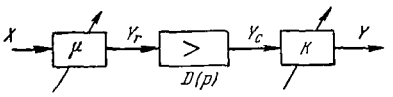

FIGURE 7.34. An element of Kochenburger's system.

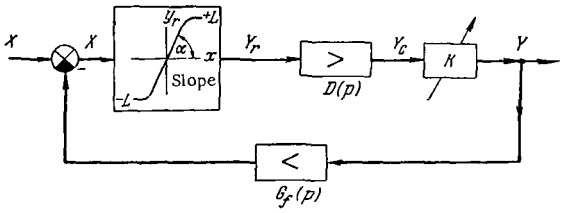

FIGURE 7.35. The oscillatory circuit in Kochenburger's system.

Kochenburger's control scheme, however, is fairly complicated. This will become the more obvious once the same problem is solved by using the methods of this chapter.

First we briefly review Kochenburger's original solution. The following convenient representation of the original system is proposed. Since the parameters of $D(p)$ are constant and only K is variable and since the controller operator $R(p)$ is also constant and only the controller gain μ is altered, $D(p)$, $R(p)$, K, and μ are represented by separate elements, as is shown in Figure 7.34, where the output signal of $R(p)$ is delivered to the input of the element with controlled μ. Figure 7.35 shows an auxiliary feedback loop which ensures the appropriate variation in μ. This circuit uses a very-high-gain amplifier with a limiter and a linear feedback element $G_f(p)$. The output signal of the high-gain amplifier is limited by a special feedback arrangement, not shown in the figure. The amplifier characteristic thus has a linear section limited between $Y_r = \pm L$. The filter of the feedback element $G_f(p)$ is so chosen that high frequency sustained oscillations are excited in the circuit for all values of the plant gain K. These oscillations provide the sampling signals in the auxiliary circuit.

The angular frequency ω_d of the oscillations should be sufficiently high, so that the amplitude of the oscillations at the output of the plant (which acts as the filter) is negligible.

The characteristic of the limiting high-gain amplifier is so chosen that when no input signal x is delivered, the amplifier has a zero average output.

Now suppose that x takes on a certain constant (say, positive) value. The output of the high-gain amplifier is biased and the average value is no longer zero. If α, L, and $G_f(p)$ are appropriately chosen, Y_r and x will be nearly proportional to each other, i.e., $Y_r \cong \mu x$. If the signal x is constant, the gain μ is proportional to the mean value of the constant output component of the high-gain amplifier.

Kochenburger has shown that the proportionality coefficient varies approximately in inverse proportion to K. The sought functional dependence for the variable gain μ is thus obtained. It is moreover shown that the results are also valid for a slowly varying x. It now remains to vary the gain μ in proportion to Y_r, so that $\mu K \cong$ const.

This method is applied to synthesize the circuit shown in Figure 7.36 for

$$D(p) = \frac{K}{(1+0.2p)^2(1+0.005p)} \tag{7.176}$$

and K varying by a factor of 1:100.

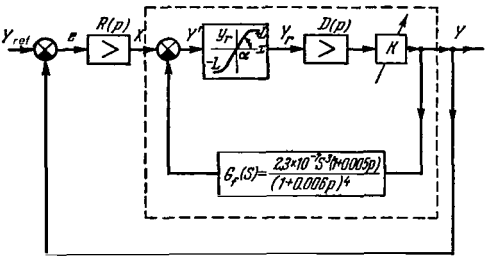

FIGURE 7.36. General configuration of Kochenburger's system.

We now solve the same problem by using the methods of this chapter. The problem is stated as follows. Find a fixed-structure system (without a searching element) which maintains the controlled variable Y constant while the plant gain varies in a ratio of 100 : 1, the plant transfer function being

$$W(p) = \frac{K}{(1+0.2p)^2(1+0.005p)} \tag{7.177}$$

(a different range of gain variation may of course be assumed).

Three linear amplifiers with a sufficiently high gain are connected in series with the controlled object. Two of these amplifiers are controlled by feedback elements $\frac{1}{T_1p+1}$ and $\frac{1}{T_2p+1}$, respectively. The entire control system takes the form shown in Figure 7.37.

We proceed to determine the system transfer function. Using the nomenclature of Figure 7.37, we write

$$\frac{Y(p)}{Y_{ref}(p)} = \frac{\dfrac{K_h}{1+\dfrac{K_h}{1+T_1p}}\dfrac{K_h}{1+\dfrac{K_h}{1+T_2p}}K_h\dfrac{K}{(1+0.2p)^2(1+0.005p)}}{1+\dfrac{K_h}{1+\dfrac{K_h}{1+T_1p}}\dfrac{K_h}{1+\dfrac{K_h}{1+T_2p}}K_h\dfrac{K}{(1+0.2p)^2(1+0.005p)}}$$

and after elementary manipulations

$$\frac{Y(p)}{Y_{ref}(p)} = \frac{K_h^3(1+T_1p)(1+T_2p)K}{(1+T_1p+K_h)(1+T_2p+K_h)(1+0.2p)^2(1+0.005p)+ \\ +K_h^3K(1+T_1p)(1+T_2p)} \qquad (7.178)$$

Dividing the numerator and the denominator by K_h^3 and putting $\frac{1}{K_h}=m$, we write

$$\frac{Y(p)}{Y_{ref}(p)} = \frac{K(1+T_1p)(1+T_2p)}{P_1}, \qquad (7.179)$$

where

$$P_1 = m^3(1+T_1p)(1+T_2p)(1+0.2p)^2(1+0.005p)+ \\ +m^2[2+(T_1+T_2)p](1+0.2p)^2(1+0.005p)+ \\ +m(1+0.2p)^2(1+0.005p)+K(1+T_1p)(1+T_2p)$$

or

$$\lim_{m\to 0}\frac{Y(p)}{Y_{ref}(p)} = 1. \qquad (7.180)$$

In other words, we succeeded in compensating the gain variation and incidentally obtained a high-quality control system. In order for the results to be realizable, the system should be tested for stability as $m \to 0$.

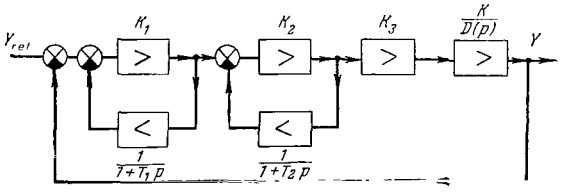

FIGURE 7.37. Illustrating the solution of Kochenburger's problem.

The characteristic equation is obtained by putting the determinant of (7.179) equal to zero, thus:

$$m^3(1+T_1p)(1+T_2p)(1+0.2p)^2(1+0.005p)+ \\ +m^2[(T_1+T_2)p+2](1+0.2p)^2(1+0.005p)+ \\ +m(1+0.2p)^2(1+0.005p)+K(1+T_1p)(1+T_2p)=0. \qquad (7.181)$$

The degree of any two successive polynomials decreases by one, and we thus have an auxiliary equation of the first kind. After some manipulations, we obtain the auxiliary equations in explicit form:

$$0.0002T_1T_2q^3+0.0002(T_1+T_2)q^2+0.0002q+KT_1T_2=0. \tag{7.182}$$

The coefficients of this equation should satisfy Hurwitz' criteria. The constraint imposed on K is

$$K<\frac{0.0002(T_1+T_2)}{T_1^2T_2^2}. \tag{7.183}$$

Hence we can readily choose the time constants that ensure stability in the entire range of gain variation; thus, for $T_1=T_2=0.01$, we have $K<400$. In other words, the gain may take on any value from zero to 400.

Kochenburger is concerned with the case of a system which can accommodate a gain increase by a factor of 100 : 1. Our stability range is much wider than that. The degenerate equation

$$(1+T_1p)(1+T_2p)=0$$

always satisfies Hurwitz' conditions. If the plant time constants are τ_1, τ_2, and τ_3, relation (7.183) takes the form

$$K<\frac{\tau_1\tau_2\tau_3(T_1+T_2)}{T_1^2T_2^2}. \tag{7.184}$$

In conclusion there is one other problem to be considered. In Kochenburger's paper it is assumed that the rate of gain variation may be comparable with the time rate of transients in the system. It is clear from our result (equation (7.180)) that for sufficiently high gain the transients are very short-lived and no additional tests are required. However, if the gain is such that the transient time constant in the system is comparable with the time rate of variation in K, the solution is valid only if the system is additionally tested for absolute stability in the given K range. This test can be readily made using V. M. Popov's method /1/.

The theoretical performance of the system shown in Figure 7.37 was tested using a model. The plant time constants were $\tau_2=0.2$ and $\tau_3=0.005$, i.e., the same values as in Kochenburger's paper. The stabilizer time constants were $T_1=T_2=0.01$ and the plant gain varied from $K_{min}=0.1$ to $K_{max}=10$. The amplifier gain was $K_h=200$.

Figure 7.38 is an oscillogram of the process for $K=10$ and Figure 7.39 is the oscillogram for $K=0.1$. Figure 7.40 is an oscillogram of a system with a sine-law gain K varying at a frequency $\omega_d=1$ cps. In all cases $Y_{ref}=1$.

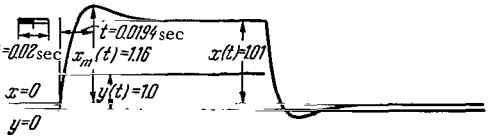

FIGURE 7.38. Oscillogram for $K=10$.

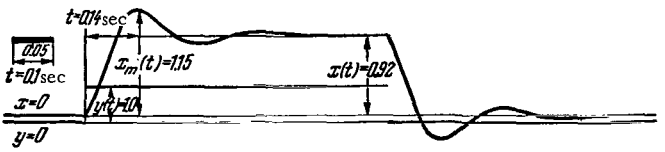

FIGURE 7.39. Oscillogram for $K = 0.1$.

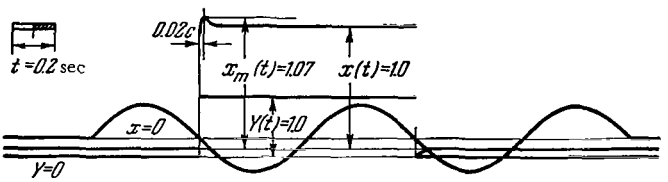

FIGURE 7.40. Oscillogram for $K = 5 - 4 \sin 6.28\, t$.

We see from these oscillograms that:

1. The steady-state value of the controlled variable is the same in all cases, i.e., the system indeed maintains the controlled variable independent of the plant gain and its variation.

2. The transient is virtually the same in all the three cases, an obviously satisfactory result.

Chapter Eight

VARIATIONAL ASPECTS OF MULTIVARIABLE CONTROL

§ 8.1. MULTIVARIABLE CONTROL AS A VARIATIONAL PROBLEM

We have noted before that multivariable control systems can be divided into two classes with fundamentally different optimization behavior. Since this division is of the utmost significance for correct choice of optimality tests and efficient design of control systems, we will go into this problem in some more detail.

In Chapter One we analyzed the problem of strip gage control in hot rolling mills and found that the quality of the metal depended on the precision with which a number of parameters were controlled, e.g., main drive speed, roll position, etc. However, improvement of the dynamics of each individual controlled variable does not necessarily mean that the system as a whole is optimized. Optimizing is attainable if the control of the individual variables is aimed from the very start at the principal target, namely achieving the necessary geometrical dimensions of the rolled strip.

Another example is provided by oil reservoirs, which were considered in Chapter One as an object of multivariable control. Efficient exploitation of the field, in the sense prescribed by our problem, is attaining maximum output (in the limit, draining the reservoir of all its oil) in the shortest possible time and at the lowest possible cost. Constant field operating conditions are maintained by sinking additional injection wells through which water is driven into the strata, and the field parameters can be regulated by adjusting the working conditions of these injection wells. Field exploitation, however, is further constrained by the large-scale requirements of national oil industry. In principle, oil fields can be worked in a multitude of different ways, while in practice the output is limited by the capacity of the equipment. Now, even if the equipment limitations have been allowed for, we are still left with a variety of well exploitation conditions and it is our job to select the optimal alternative.

The oil-and-water-bearing strata in conjunction with the well constitute a single hydrodynamical system. If the outputs of some of the wells are altered, pressures and flow patterns in the entire field are affected. For example, enhanced exploitation of a number of wells only, with continued injection of water, will eventually lower the stratal pressure, and many wells may stop producing; moreover, formation water may penetrate into the region of reduced pressure, and some wells will be prematurely flooded.

Well depletion and flooding raise the cost of field exploitation. It is clear, therefore, that the well operating conditions should be chosen with due consideration to economic factors.

One of the fundamental requirements in planning the well pattern is that the formation pressure distribution comply with the desirable working conditions and, in particular, the possibility of natural flow. This of course imposes additional restrictions on oil field exploitation, and it is by no means certain that the result is the optimum. The point is, that the real oil reservoir is inhomogeneous in its physical and chemical properties, so that each well has a different potential. Moreover, starting at a certain stage of oil field exploitation, the wells are all flooded in varying degrees. The flooding is generally more pronounced in wells with high production rates. It should therefore be understood that driving a well at a maximum rate of production may eventually lower the output and increase the production costs for the field. On the other hand, hopelessly flooded wells can be "suppressed" (or even discontinued entirely), so that the total oil output increases markedly. We are thus clearly faced with a variational problem of optimizing the oil production conditions under a given set of constraints.

In the two cases above, multivariable control provides an adequate solution of the problem, and each individual variable should be controlled in such a way as to extremize some generalized quality index of the system as a whole. In this chapter we will consider the detailed solution of the problem in application to the simplest case of oil field exploitation.

All the preceding refers to systems of one class. The other class includes multivariable control systems which are optimized by optimizing every individual controlled variable. We will show in the following that in this case also the control equation is obtained by solving a variational problem.

§ 8.2. APPLICATION OF LINEAR PROGRAMMING

The linear programming (LP) problem can be stated as follows: find a vector $\mathbf{y}(y_1, \ldots, y_n)$ maximizing (minimizing) the linear form

$$R = \sum_{i=1}^{n} c_i y_i, \tag{8.1}$$

where the variables satisfy the linear constraints

$$y_j \geqslant 0 \qquad (j = 1, 2, \ldots, n) \tag{8.2}$$

and

$$\left.\begin{aligned} &a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n = b_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{j1}y_1 + a_{j2}y_2 + \ldots + a_{ji}y_i + \ldots + a_{jn}y_n = b_j, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mi}y_i + \ldots + a_{mn}y_n = b_m, \end{aligned}\right\} \tag{8.3}$$

where a_{ij}, b_i, and c_i are known constants and $m < n$.

In matrix form the general LP problem is written as follows: maximize (minimize)

$$CY \tag{8.4}$$

subject to the condition

$$Y > 0, \qquad AY = b. \tag{8.5}$$

Here C is a row matrix, Y a column matrix, $A=[a_{ij}]$ a $m\times n$ matrix, B a column matrix.

In Chapter One we derived a set of algebraic equations which, under certain conditions, approximately describe the behavior of an oil field. These equations are based on the assumption of linear seepage (Darcy's linear law of filtration) and rigid operating conditions. The relation between debit and pressure in a well is given by the expression (see Chapter One)

$$AQ = \Delta P, \tag{8.6}$$

where A is a regular $m\times n$ matrix, Q and ΔP are n-component column matrices. The elements of the matrix A are found from the relations

$$a_{ij} = \frac{\Delta P_i}{\Delta Q_j} \tag{8.7}$$

For n producing wells the matrix equation (8.6) is a set of n linearly independent equations which for $Q_j \geqslant 0$ define the boundary of a closed convex polyhedron. If P_i and Q_i are varied, the hodograph of the vector Q_i will fill a certain domain containing all the points of the convex set. The matrix equation (8.6) in this case may be clearly given by the inequality

$$AQ \leqslant \Delta P$$

or, alternatively,

$$\sum_{j=1}^{n} a_{ij}Q_j \leqslant \Delta P_i \tag{8.8}$$
$$(i = 1, 2, \ldots, n).$$

The preceding discussion also has a vivid geometrical interpretation. As an example, consider production from two wells. Relations (8.8) take the form

$$\left.\begin{aligned} a_{11}Q_1^p + a_{12}Q_2^p \leqslant P_k - P_1^p, \\ a_{21}Q_1^p + a_{22}Q_2^p \leqslant P_k - P_2^p. \end{aligned}\right\} \tag{8.9}$$

Here p denotes the producing well, k the pressure on the field boundary; $P_{1,2}$ is the well pressure. Geometrically (8.9) describes a convex quadrangle $OABC$ (Figure 8.1), which is obtained in the following way. First put $Q_1>0$ and $Q_2>0$. We are thus concerned only with the first quadrant of the Q_1,Q_2 plane, limited by $Q_1=0$ and $Q_2=0$.

Now consider where the first inequality of (8.9) reduces to an equality,

$$a_{11}Q_1^p + a_{12}Q_2^p = P_k - P_1^p. \tag{8.10}$$

The half-plane containing all the solutions of inequality (8.9) (the first inequality) is located below and to the left of the line (8.10). The equation of this line is

$$Q_2 = -\frac{a_{11}}{a_{12}} Q_1^p + \frac{P_k - P_1^p}{a_{12}}. \tag{8.11}$$

The intercept of this line on the Q_2 axis is

$$\frac{P_k - P_1^p}{a_{12}} \tag{8.12}$$

and its intercept on the Q_1 axis is

$$\frac{P_k - P_1^p}{a_{11}}. \tag{8.13}$$

The second line is constructed in the same way. We thus delineate a region where expression (8.6) holds true.

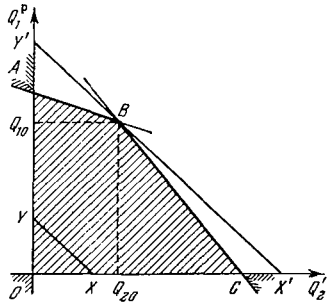

FIGURE 8.1. Illustrating the linear programming problem.

Now consider the linear form

$$v = Q_1 + Q_2, \tag{8.14}$$

which gives the total output (water and oil) of the two wells. Let us maximize the total output. This is best done by a geometrical construction. For particular values of v expression (8.14) describes a family of straight lines which are marked in Figure 8.1 as XY. The maximum is attained at the point Q_{10}, Q_{20} where the line XY is tangent to the convex quadrangle (point B in Figure 8.1). The output is thus maximum when well 1 produces Q_{10} and well 2 produces Q_{20}. This result holds true if the only constraints are those imposed by the inherent properties of the oil reservoir. In what follows we call these constraints the technological constraints of the variational problem.

Linear programming is thus applicable to optimizing multivariable control systems described by linear algebraic equations with a generalized quality criterion, which is a linear form in the controlled variables.

§ 8.3. THE PROBLEM OF OPTIMUM OIL-FIELD EXPLOITATION

The applicability of linear programming to oil production optimizing was illustrated for the simple case of two producing wells. Let us now consider a more general case often encountered in practice.

We have already discussed some technological constraints. The main parameters to be constrained are the permissible and the maximum formation and well pressures. In practice ΔP can be arrived at by considering the permissible and the maximum well pressures. The various requirements of the production schedule for the different parts of the field and the redistribution of flow streams needed to control formation water circulation can be satisfied by forming linear combinations of some components ΔP_i.

However, optimum production schedule depends not only on the interrelationship between wells and the maximum pressures in production and injection lines. Another class of restrictions are connected with the limited capacity of equipment:

$$\sum_j Q_j \leqslant Q \qquad (j = m, \ldots, l). \tag{8.15}$$

In what follows constraints (8.15) will be regarded as the production constraints of the variational problem. Relations (8.15) correspond, e.g., to pumping restrictions associated with the productivity of water-disposal equipment or intermediate water-pumping stations. Similar inequalities may represent production restrictions because of insufficient through capacity of demulsifying plants, storage pools, gravity-flow and head-flow collectors.

There are also restrictions of purely economic character. The majority of economic constraints are associated with capital outlay. In well optimizing it is assumed that the plant (i.e., the number of wells and the well pattern) is given. The capital investment may therefore be regarded as constant during a certain period of time. Since the investment does not change, the economic constraints are ignored at this stage.

The choice of the optimum well pattern is a complicated problem of independent interest, and we will not go into it here.

Consider the field exploitation charges, which can be itemized as a function of well outputs. The power requirements can be written as a linear function of the outputs, thus:

$$\sum_j b_j Q_j \leqslant N, \tag{8.16}$$

where b_j are the charge coefficients, N the power restrictions.

Production planning criteria impose additional constraints of the form

$$\sum_{i=1}^{n} C_i Q_i \geqslant Q_t, \tag{8.17}$$

where C_i is the proportion of oil in the fluid lifted from the i-th well, Q_t the oil production target.

The variational problem can now be stated in two alternative forms:

(1) Find well operating conditions ensuring maximum total oil output under given technological, production, and economic constraints.

(2) Find well operating conditions ensuring minimum production cost for the planned output under given technological and production constraints.

In principle, the two statements are identical. Therefore, without loss of generality, we will only consider the problem of maximum total output under given constraints, where the functional (the object function) is written in the form

$$F(Q)=\sum_{i=1}^{n} C_i Q. \tag{8.18}$$

The set of equations specifying the technological constraints are thus combined with expressions for production and economic constraints. If the combined constraint matrix is designated K_{ij}, we arrive at the following statement of the variational problem:

Optimize

$$\sum_i C_i Q_i \tag{8.19}$$

given

$$\|K_{ij}\| \|Q_j\| \leqslant \|\Gamma\| \tag{8.20}$$

and

$$Q_j > 0. \tag{8.21}$$

Here $\| K_{ij} \|$ is a rectangular $m \times n$ matrix with $m>n$; $\| \Gamma \|$ is the m-component constraint vector (column matrix).

A few words about the coefficient C_i. It is defined as the proportion of oil in the pumped fluid: $C_i=1$ indicates that the well produces pure oil, whereas $C_j=0$ means that the fluid contains no oil altogether (as is the case in injection wells, say).

§ 8.4. A NUMERICAL EXAMPLE

The theory of the preceding section can be illustrated by a numerical example. The data below do not apply to any particular reservoir, but they are nevertheless typical. Consider a sector with six wells. Figure 8.2 shows the well pattern and the formation boundaries; the numerical values of the hydraulic resistance are also given. This information is sufficient to write the equation of linear seepage. The figures were obtained from (8.7) using a grid model.

Wells 1 and 6 are injection wells, so that C_1 and C_6 are both zero (the fluid is pure water). Wells 2, 3, 4, 5 are producing wells with mechanical sucker-rod pumping, electrical centrifugal pumping, natural flow, and hydraulic long-stroke pumping, respectively. The proportion of oil in the fluid lifted from these wells is respectively 0.4, 0.9, 1, and 0.1.

The maximum output differs from well to well depending on productivity coefficients, formation and well pressures, and also the layout of auxiliary

TABLE 8.1

	Specifications	Constraints	Justification
1	Injection well pressure not to exceed the allowed maximum of the equipment	$P_i \leqslant [P_i]$	Direct experiment with the water pumps connected to one of the wells
2	Producing well pressure not below a certain limiting figure	$P_i \geqslant [P_i]$	Calculations based on pump stroke length and minimum self-flowing pressures
3	To ensure stable natural flow, the dynamic formation pressure for well 4 not to be less than a certain limiting figure	$P_4 \mid_{Q_4=0} \geqslant [P_4]$	Calculations based on item 2 under given output restriction
4	To prevent gas invasion, the dynamic formation pressure for well 3 not to be less than saturation pressure	$P_3 \mid_{Q_3=0} \geqslant P_s$	The equipment will lift at well pressures below saturation pressure
5	To avoid premature flooding of well 4, the water tongue in the dangerous direction 3—4 should be tied to well 3	$Q_3 \geqslant Q_4$	Analysis of depression regions on the grid model
6	Siphon output and secondary recovery water pump output not to exceed certain limiting figures	$Q_s \leqslant [Q_s]$ $Q_r \leqslant [Q_r]$	Direct experiment
7	Through capacity of the well 3-to-well 4 gravity-flow collector not to exceed a certain limiting figure	$Q_g \leqslant [Q_g]$	Calculations and direct experiment
8	Demulsifying plant productivity not to exceed a certain limiting figure	$Q_d \leqslant [Q_d]$	Direct experiment and statistical data
9	Product released from storage pool to meet certain quality standards	$\eta \leqslant [\eta]$	Consumer requirements
10	Power requirements not to exceed a prescribed figure	$N \leqslant [N]$	The statistical dependence $N=N(Q)$ is derived empirically

TABLE 8.2

Output	Well number					
	1	2	3	4	5	6
Q_i	118.5	100	25	25	0	12.5
C_iQ_i	0	40	22.5	25	0	0

storage and processing installations. Figure 8.3 shows the general layout and the communication lines. It is assumed that the technological, production, and economic constraints are all known. The relevant information is listed in Table 8.1.

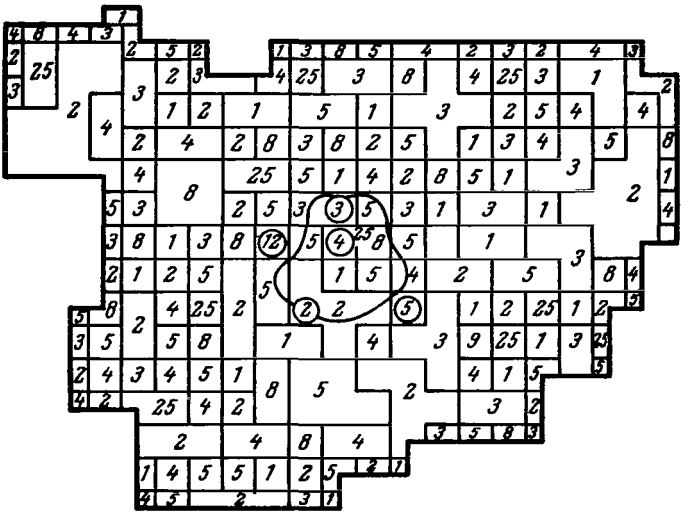

FIGURE 8.2. Oil reservoir data.

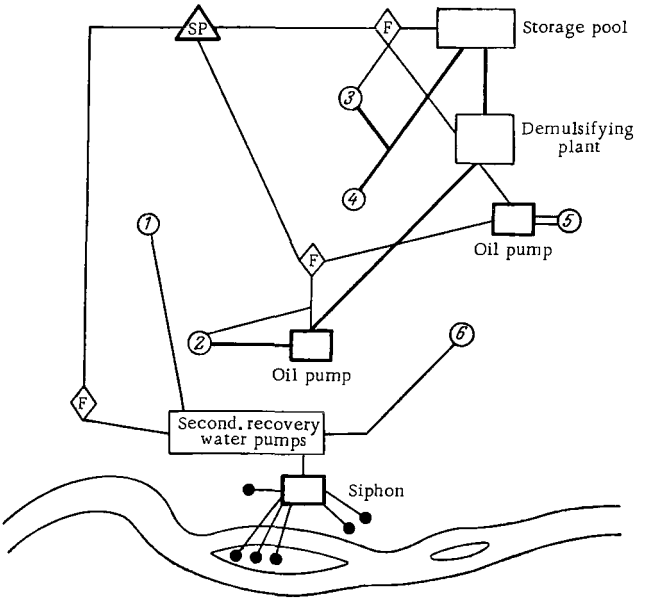

FIGURE 8.3. Layout of oil-field installations.

The problem is to find well operating conditions that ensure maximum oil production under the given constraints. Using Table 8.1 and the numerical data, we formulate the following mathematical problem.

Maximize the linear form

$$\|0,\ 0.4,\ 0.9,\ 1,\ 0.1,\ 0\| \begin{Vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Vmatrix} \tag{8.22}$$

under the given constraints

$$\begin{Vmatrix} 072 & -011 & -011 & -012 & -006 & 006 \\ -011 & 092 & 008 & 011 & 006 & -009 \\ -011 & 008 & 235 & 019 & 010 & -012 \\ -012 & 011 & 019 & 555 & 011 & -012 \\ -006 & 006 & 010 & 011 & 181 & -011 \\ 006 & -009 & -008 & -012 & -011 & 043 \\ -000 & 000 & -100 & 100 & 100 & 000 \\ -011 & 008 & 000 & 019 & 010 & -008 \\ -012 & 011 & 019 & 000 & 011 & -012 \\ 100 & 000 & 000 & 000 & 000 & 100 \\ 000 & 000 & 100 & 100 & 000 & 000 \\ 000 & 100 & 000 & 000 & 100 & 000 \\ 000 & -040 & 100 & -100 & 000 & 000 \\ 020 & 005 & 007 & 000 & 000 & 020 \end{Vmatrix} \begin{Vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Vmatrix} \leqslant \begin{Vmatrix} 70 \\ 105 \\ 140 \\ 120 \\ 70 \\ 50 \\ 0 \\ 110 \\ 0 \\ 250 \\ 50 \\ 100 \\ 0 \\ 95 \end{Vmatrix}$$

The problem was solved by the simplex method, and the results are listed in Table 8.2.

§ 8.5 SOME GENERAL CONSIDERATIONS

We should first justify the application of linear programming to the oil field exploitation problem. In the general case of a plant without memory with constant coefficients in the algebraic equations describing its behavior, linear programming can be used to a considerable advantage, especially since the numerical algorithms of this method are easily adapted to digital computers. Straightforward application of linear programming to oil production control, however, would be somewhat improper, since no real oil reservoir is actually maintained under steady-state conditions. The coefficients a_{ij} entering the initial equations of the technological process and the constraint inequalities are variable in time, and not constant. In some cases the coefficients a_{ij} change very slowly and gradually (e.g., in the case of migration of the formation boundary), whereas sometimes they will change abruptly (as when the target figures are modified).

The resulting difficulties can be overcome if the coefficients are adjusted as we go on, to meet the change in standards and specifications. The main difficulties thus arise due to the requirement of systematic

adjustment of the coefficients a_{ij}, which are dependent on the state of the oil reservoir. The values of these coefficients can be determined in practice only by using each well successively to introduce a certain disturbance into the process, while decoupling all the other wells, whose operation is stabilized with respect to the disturbance parameter.

Such operating conditions can be achieved by automatic stabilization of well operation. However, the direct experimental approach does not appear particularly promising in view of the exceedingly slow transient in the well-formation system and the rapid reduction of coupling with the distance from the source of the disturbance. Figure 8.4a plots the pressure recovery in a well distant 500 m from the disturbing well, which stopped producing at $t=0$. Before that, the stopped well had operated for a long time with constant output /40/. We see from the curve that direct experimental determination of a_{ij} requires well observations over a number of months. Besides being impracticable, this approach is inadequate since during such a long period other formation parameters may also change appreciably.

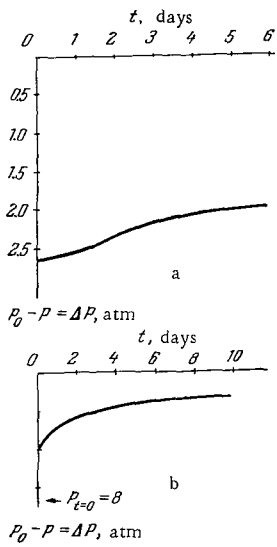

FIGURE 8.4. Transients in an oil-bearing formation.

The coefficients a_{ij} can be obtained by direct experiment only if special well stabilizing systems are provided (see Figure 8.4b). In this case a model of the controlled object is incorporated in the control system and updated at fixed intervals on the basis of current information on formation structure obtained by geological, geophysical, and hydrodynamic methods /72, 75/. Successful models have been actually devised for more or less uniform formations /72, 75/, but no adequate grid models have been built for the general case of a reservoir of complex structure. The main difficulties are associated with the determination of the mathematical nonuniformity function of the formation.

The control system described in the preceding section is suitable for homogeneous or quasihomogeneous formations, where the structure of the producing strata is such that the distribution of inhomogeneities between any two nearby wells is constant or follows the distribution of stationary random events.

The flow chart of a control system of this kind is shown in Figure 8.5. The basic elements are the grid model used to determine the corresponding coefficients a_{ij} and a digital computer that calculates production schedules for each well.

The difficulties associated with the slow variation of the well coupling coefficients are overcome by periodically updating the position of the oil-water boundary on the grid model.

As the process drifts from the optimum or when the target figures are changed, the entire closed-loop control system is turned on (Figure 8.5). When the wells have been restored to the desired operating mode, the computers are disconnected and only the local control systems and the

data-processing system are left running. The local control systems maintain the given operating mode in the interval between successive adjustments, and the information received from the wells through the data-processing system and through other channels is used to update the grid model. When the need arises, the computers are again linked into the system, and the entire cycle is repeated. Statistical forecasting techniques can be used to calculate the coefficients a_{ij}.

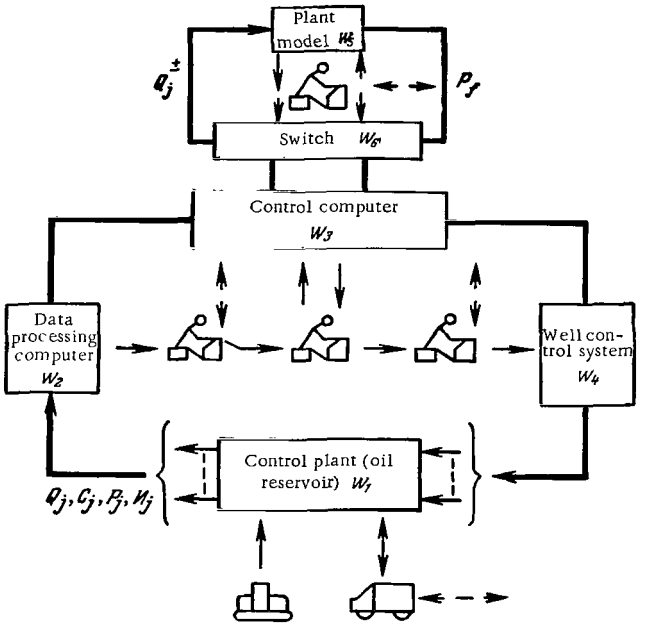

FIGURE 8.5. Flow chart of automatic control system

The solution of the problem is based on the assumption of a rigid operating mode (see Chapter One). It has been established, however, that immediately following the disturbance (when a well is stopped or actuated, etc.), the oil reservoir behaves according to a so-called elastic mode /75/. Although the processes in the reservoir may remain linear in the sense that the principle of superposition holds true, linear programming in its standard form is inapplicable. It is therefore again emphasized that our solution is valid for a reservoir in a rigid operating mode, which is the predominant but not the only mode.

§ 8.6. METHODS FOR THE DETERMINATION OF THE CONTROL VECTOR AS A FUNCTION OF TIME IN MORE COMPLICATED CASES

In the preceding sections linear programming was used to determine the operation schedule for each oil well. This approach is valid as long

as the controlled object (e.g., the oil-bearing formation, ignoring its elastic properties) is described by a set of linear algebraic equations. The planning and production constraints were represented by appropriate algebraic equations, and the solution was obtained in the form of a numerical programme, or schedule, for each well.

For objects with memory the control function cannot be obtained in this simple form, but the problem is nevertheless meaningful for some more complex cases. Let us first consider the new features arising from the formulation adopted in this section, although the problem itself is basically equivalent to that considered in the previous sections. All the properties of the controlled object and all the constraints are known; we seek a control function, i.e., the variation of the inputs as a function of time, that minimizes (or maximizes) a certain criterion function. The control time is often chosen as the criterion. The problem thus reduces to a selection of a control function which ensures a minimum transient time for the given plant under the given constraints.

The optimal solution in this case is to choose, from among the control functions satisfying the given constraints, one which moves the system from the initial to the final state in a minimum time. This formulation is not different in principle from that used for most optimal control problems /21, 53/. The literature on the subject, however, is mainly confined to single-variable systems /17, 25, 28/. P. E. Sarachik and G. M. Kranc /21/ solved the problem of minimum transient time for multivariable control systems, but their solution is based on the results of Krasovskii /25/, Kirillova /17/, and Kulikowski /28/, originally obtained for single-variable systems.

A remarkable feature of the above studies /17, 25, 28, 21/ is that the problem of optimum control is solved by methods of functional analysis. In our opinion, the application of functional analysis may prove to be highly promising, and we therefore reproduce the results of Sarachik and Kranc /21/ in some detail.

We are dealing with time optimal control of an absolutely controllable linear object with certain constraints /21/. Different constraints may be imposed on each input.

The controlled object is described by the following differential equation:

$$\boldsymbol{x} = F(t)\,\boldsymbol{x}(t) + D(t)\,\boldsymbol{u}(t), \tag{8.23}$$

where $\boldsymbol{x}(t)$ is the n-dimensional state vector of the object at the time t, $\boldsymbol{u}(t)$ is the r-dimensional control vector, $F(t)$ is a $n \times n$ matrix, $D(t)$ is a $n \times r$ matrix. It is clear from this notation that the plant has r inputs and n outputs. In general, the system output is a vector $\boldsymbol{y}(t)$ related to the plant inputs by the equation

$$\boldsymbol{y}(t) = \mu(t)\,\boldsymbol{x}(t), \tag{8.24}$$

where $\mu(t)$ is a $m \times n$ matrix.

The initial state of the plant at the time $t = t_0$ is described by

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{8.25}$$

By $\mathbf{y}_{\text{ref}}(t)$ we denote the signal to be reproduced. The constraints imposed on the plant inputs $u(t)$ are given by

$$\|u_i\|_{pi} = \left[\int_{t_0}^{t_1} |u_i(\tau)|^{p_i} d\tau\right]^{\frac{1}{p_i}} \leqslant L_i, \tag{8.26}$$

where

$$p_i > 1 \text{ and } i = 1, 2, \ldots, r.$$

It is significant that p_i and L_i may be different for each input, i.e., for each i. This means that the constraints on the components of the input vector depend on the inputs themselves; thus, for an amplitude-limited input $p_i = 1$, and for a power-limited input (some other u_j) $p_j = 2$, etc.

The problem is thus formulated as follows. Find an input $\boldsymbol{u}_0(t)$ which satisfies constraints (8.26) and ensures equality of the output signal $\boldsymbol{y}(t)$ to the setting $\boldsymbol{y}_{\text{ref}}(t)$ at the time $t = t_1$, so that $T = t_1 - t \geqslant 0$ is minimum.

Following N. N. Krasovskii /25/ we can give an alternative formulation of this problem. Given the plant equations, constraints, and initial conditions, find a control vector $\boldsymbol{u}(t)$ of minimum norm $\| \boldsymbol{u} \|_p$ which ensures the equality $\boldsymbol{y}(t_1) = \boldsymbol{y}_{\text{ref}}(t_1)$ in a predetermined time t_1. The solution ensuring the fastest response is then determined from the solution of this problem, and the minimum time corresponds to the case when the minimum norm $\| \boldsymbol{u} \|_p$ is exactly 1 /25/.

Using the Duhamel integral approach, we write the solution of the set of differential equations (8.23) in the form

$$\boldsymbol{x}(t) = \Phi(t, t_0)\, \boldsymbol{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)\, D(\tau)\, \boldsymbol{u}(\tau)\, d\tau, \tag{8.27}$$

where $\Phi(t, \tau)$ is the fundamental matrix of the plant equations satisfying the condition $\Phi(t, \tau) = E$. In /21/ it is called the transition matrix.

The output signal $\boldsymbol{y}(t)$ is thus given by

$$\boldsymbol{y}(t) = \mu(t, t_0)\, \Phi(t, t_0)\, \boldsymbol{x}_0 + \int_{t_0}^{t} \mu(t)\, \Phi(t, \tau)\, D(t)\, \boldsymbol{u}(\tau)\, d\tau. \tag{8.28}$$

Equation (8.28) can be simplified in the following way. Let

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \mu(t)\, \Phi(t, t_0)\, x_0 \tag{8.29}$$

and

$$H(t, \tau) = \mu(t)\, \Phi(t, \tau)\, D(\tau), \tag{8.30}$$

where $H(t, \tau)$ is the $m \times r$ matrix of weight functions of the controlled variables in all the channels; $\boldsymbol{e}(t)$ is the difference between the actual output signal and the output signal caused by the initial disturbance alone, when no input is received for $t > t_0$. In this notation equation (8.28) is written as

$$\boldsymbol{e}(t) = \int_{t_0}^{t} H(t, \tau)\, \boldsymbol{u}(\tau)\, d\tau. \tag{8.31}$$

To solve the problem, we thus have to find the control vector $\boldsymbol{u}(\tau)$ with a minimum norm $\| \boldsymbol{u} \|_p$ satisfying the integral equation

$$\int_{t_0}^{t_1} H(t_1, \tau)\, \boldsymbol{u}(\tau)\, d\tau = \boldsymbol{e}_{\text{ref}}(t_1), \tag{8.32}$$

where

$$e_{\text{ref}}(t_1) = y_{\text{ref}}(t_1) - \mu(t_1)\Phi(t, t_0)x_0.$$

The vector equation (8.32) can be replaced by m component equations

$$\int_{t_0}^{t_1} h_j(t_1, \tau)u(\tau)d\tau = e_{\text{ref}\,j}(t_1) \quad (j = 1, 2, \ldots, m). \tag{8.33}$$

Here $h_j(t, \tau)$ is the j-th row of the matrix $H(t, \tau)$, and $e_{\text{ref}\,j}(t_1)$ is the j-th element of the vector $e_d(t_1)$. Since the object is absolutely controllable, there is at least one control vector satisfying the above conditions at an arbitrary time t_1, and the vector with the minimum norm $\|u\|_p$ should be selected from among these alternatives.

Let the set of control vectors satisfying (8.33) be u_A. Consider the functional

$$f_A(h_j) \equiv \int_{t_0}^{t_1} h_i(t_1, \tau)u(\tau)d\tau, \tag{8.34}$$

where $u(\tau)$ is an element of u_A. Hence,

$$f_A(h_j) = e_{\text{ref}\,j}(t_1). \tag{8.35}$$

Since f_A is a linear functional, we will consider arbitrary linear combinations of $h_j(t, \tau)$ of the form

$$k(t_1\tau) = \sum_{j=1}^{m} \lambda_j h_j(t_1\tau) = \lambda H(t_1, \tau), \tag{8.36}$$

where λ is an m-dimensional row vector, so that applying (8.35) and (8.36) we write

$$f_A(k) = f_A\left(\sum_{j=1}^{m} \lambda_j h_j\right) = \sum_{j=1}^{m} \lambda_j f_A(h_j) = \lambda e_{\text{ref}}(t_1); \tag{8.37}$$

this equality holds true for any $k(t_1\tau)$ described by an equation of the type (8.36).

Further solution of the problem is associated with the concept of the general norm of a vector /21/. How are we to write the set of r constraints of the form (8.26) as a single constraint? This can be done by defining the norm of the control vector u as

$$\|u\| = \max_i [\|u_i\|_{pi}/h_i]. \tag{8.38}$$

Now, if the single condition

$$\|u\| \leqslant 1 \tag{8.39}$$

is satisfied, all the r inequalities (8.26) are fulfilled, so that the single condition (8.38) is in effect equivalent to r constraints.

Since relations (8.38) are not very useful in their original form, the solution can be simplified by further generalizing the definition of the

norm of the vector $\boldsymbol{u}(t)$. The most general norm is defined as

$$\|\boldsymbol{u}\|_p = \left[\sum_{i=1}^{r} h_i^{-p} \|u_i\|_{pi}^p\right]^{1/p}, \tag{8.40}$$

where $p \geqslant 1$ and $\|u_i\|_{pi}$ are given by (8.26). The results of Kirillova /17/ can be used to show that for $p \to \infty$ the solution of the problem with a bounded $\|\boldsymbol{u}\|_p$ approaches the solution with a bounded $\|\boldsymbol{u}\|$, so that constraints (8.26) are replaced by the single inequality

$$\|\boldsymbol{u}\|_p \leqslant 1. \tag{8.41}$$

The solution of the original problem is thus obtained by first solving the problem with constraint (8.41) and then letting $p \to \infty$.

We now return to the solution of our problem. Using (8.26) and (8.41), we obtain

$$\|k_i\|_{qi} = \left[\int_0^{t_1} |k_i(t_1, \tau)|^{q_i} d\tau\right]^{1/q_i} \tag{8.42}$$

and

$$\|\boldsymbol{k}\| = \left[\sum_{i=1}^{r} h_i^q \|k_i\|_{q_i}^q\right]^{1/q}, \tag{8.43}$$

where $k_i(t_1, \tau)$ is the i-th element of the row-vector $\boldsymbol{k}(t_1\tau)$ defined by (8.36), and q_i, q are related to p_i, p from (8.40) by the equalities

$$\frac{1}{q_i} + \frac{1}{p_i} = 1$$

and

$$\frac{1}{q} + \frac{1}{p} = 1.$$

Consider the quantity $\|f_A\|$, the so-called norm of the functional f_A /28/, defined by

$$\|f_A\| = \max_{\boldsymbol{k}} \left\{\frac{f_A(\boldsymbol{k})}{\|\boldsymbol{k}\|_q}\right\}, \tag{8.44}$$

where $\boldsymbol{k}$ is from (8.36). Using (8.36), (8.37), and (8.44), we find

$$\|f_A\| = \max_{\boldsymbol{\lambda}} \left\{\frac{|\boldsymbol{\lambda} l_{\text{ref}}(t_1)|}{\left\|\sum_{j=1}^{m} \lambda_j \eta_j\right\|_q}\right\} = \max_{\boldsymbol{\lambda} l_{\text{ref}} = 1} \left\{\frac{1}{\left\|\sum_{j=1}^{m} \lambda_j \eta_j\right\|_q}\right\}. \tag{8.45}$$

Let $\boldsymbol{\lambda}^*$ be the vector whose coordinates minimize the norm $\left\|\sum_{j=1}^{m} \lambda_j \eta\right\|_q$ on condition $\boldsymbol{\lambda} e_{\text{ref}}(t_1) = 1$. Then

$$\min_{\boldsymbol{\lambda} e_{\text{ref}} = 1} \left\|\sum_{j=1}^{m} \lambda_j \eta_j\right\|_q = \left\|\sum_{j=1}^{m} \lambda_j^* \eta_j\right\|_q = \|\boldsymbol{k}^*\|_q. \tag{8.46}$$

Under these conditions equation (8.37) may be written as

$$\|p_A\| = \frac{1}{\|k^*\|_q}. \tag{8.47}$$

To proceed further, we require the generalized Hoelder inequality. Hoelder's inequality for sums is /33/

$$\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \leqslant \left[\sum_{i=1}^{n} \alpha_i^p\right]^{1/p} \left[\sum_{i=1}^{n} \beta_i^q\right]^{1/q}, \tag{8.48}$$

and the equality is obtained if and only if

$$\alpha_i = k|\beta_i|^{q-1} \operatorname{sign} \beta_i \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{8.49}$$

k being an arbitrary constant. We are interested in Hoelder's inequality for the integral

$$\int_a^b x(t)\, y(t)\, dt = \int_a^b \sum_{i=1}^{n} x_i(t)\, y(t)\, dt. \tag{8.50}$$

We have to prove that

$$\left|\int_a^b x(t)\, y(t)\, dt\right| \leqslant \left[\sum_{i=1}^{n} L_i^{-p} \|x_i\|_{p_i}^p\right]^{1/p} \left[\sum_{i=1}^{n} L_i^q \|y_i\|_{q_i}^q\right]^{1/q}, \tag{8.51}$$

where

$$p \geqslant 1, \quad q \geqslant 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1;$$

$|x_i(t)|^{p_i}$ and $|y_i(t)|^{q_i}$ are integrable, and L_i are positive quantities. Moreover,

$$\|x_i\|_{p_i} = \left[\int_a^b |x_i(t)|^{p_i} dt\right]^{1/p_i} \tag{8.52}$$

and

$$\|y_i\|_{q_i} = \left[\int_a^b |y_i(t)|^{q_i} dt\right]^{1/q_i}$$

for

$$p_i \geqslant 1, \quad q_i \geqslant 1, \quad \frac{1}{p_i} + \frac{1}{q_i} = 1.$$

We first prove that inequality (8.51) reduces to an equality if and only if

$$x_i(t) = kL_i^q \|y_i\|_{q_i}^{(q-q_i)} |y_i(t)|^{q_i - 1} \operatorname{sign} y_i(t) \tag{8.53}$$

for any $a \leqslant t \leqslant b$, any $i = 1, \ldots, n$, and an arbitrary constant k.

Hoelder's integral inequality is generally written in the form

$$\left|\int_a^b x(t)\, y(t)\, dt\right| \leqslant \left[\int_a^b |x(t)|^p\, dt\right]^{1/p} \left[\int_a^b |y(t)|^q\, dt\right]^{1/q} \tag{8.54}$$

with $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and integrable $|x(t)|^p$ and $|y(t)|^q$; the equality is obtained if and only if

$$x(t)=k|y(t)|^{q-1}\operatorname{sign} y(t), \tag{8.55}$$

k being an arbitrary constant.

It has been proved /21/ that these results hold true for $p=1$ and $q=1$. It has been further established /21/ that Hoelder's inequality for sums (8.48) and condition (8.49) hold true for $p=1$ and $q=1$. To obtain Hoelder's inequality for the integral (8.50), we note that

$$\left|\int_a^b \sum_{i=1}^n x_i(t)\,y_i(t)\,dt\right| \leqslant \sum_{i=1}^n \left|\int_a^b x_i(t)\,y_i(t)\,dt\right|, \tag{8.56}$$

and the equality is obtained if and only if

$$x_i(t)\,y_i(t)\geqslant 0 \quad [\text{or } x_i(t)\,y_i(t)\leqslant 0] \tag{8.57}$$

for $a\leqslant t\leqslant b$ and $i=1, \ldots, n$.

Using Hoelder's inequality in the form (8.54), we find that

$$\left|\int_a^b x_i(t)\,y_i(t)\,dt\right| \leqslant \|x_i\|_{p_i}\|y_i\|_{q_i} \tag{8.58}$$

for $p_i\geqslant 1$, $q_i\geqslant 1$ and $\frac{1}{p_i}+\frac{1}{q_i}=1$, where $|x_i(t)|_{p_i}$ and $|y_i(t)|_{q_i}$ are integrable. Using (8.55) we find that (8.58) reduces to an equality only if

$$x_i(t)=K_i|y_i(t)|^{q_i}\operatorname{sign} y_i(t) \quad \text{for} \quad a\leqslant t\leqslant b. \tag{8.59}$$

Substituting (8.50) and (8.58) in (8.56), we find

$$\left|\int_a^b x(t)\,y(t)\,dt\right| \leqslant \sum_{i=1}^n \|x_i\|_{p_i}\|y_i\|_{q_i}. \tag{8.60}$$

If (8.59) holds true for any $i=1, \ldots, n$ and if all K_i have the same sign, condition (8.57) is satisfied. This means that the equality in (8.60) is obtained if and only if (8.59) holds true for any $i=1, \ldots, n$ and all K_i have the same sign.

Let

$$\hat{x}_i=\frac{\|x_i\|_{p_i}}{L_i}$$

and (8.61)

$$\hat{y}_i=L_i\|y_i\|_{q_i},$$

where L_i is a positive constant.

$\hat{x}_i$ and $\hat{y}_i$ being positive, we have

$$\sum_{i=1}^n \hat{x}_i\hat{y}_i=\left|\sum_{i=1}^n \hat{x}_i\hat{y}_i\right|. \tag{8.62}$$

Using (8.48) and (8.49), we find

$$\left|\sum_{i=1}^{n}\hat{x}_i\hat{y}_i\right| \leqslant \left[\sum_{i=1}^{n}|\hat{x}_i|^p\right]^{1/p}\left[\sum_{i=1}^{n}|\hat{y}_i|^q\right]^{1/q}, \tag{8.63}$$

where $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q}=1$, and the equality in (8.63) is obtained if and only if

$$\hat{x}_i = \hat{k}\,|\hat{y}_i|^{q_i-1}\operatorname{sign}\hat{y}_i \tag{8.64}$$

for all i, k being a positive number. Since $\hat{x}_i$ and $\hat{y}_i$ are positive, condition (8.64) takes the form

$$\hat{x}_i = k\hat{y}_i^{q-1}. \tag{8.65}$$

Substituting (8.61) in (8.63) and making use of (8.60), we obtain Hoelder's generalized inequality in the following final form:

$$\left|\int_a^b \boldsymbol{x}(t)\,\boldsymbol{y}(t)\,dt\right| \leqslant \left[\sum_{i=1}^{n} L_i^{-p}\|x_i\|_{p_i}^{p}\right]^{1/p}\left[\sum_{i=1}^{n} L_i^{q}\|y_i\|_{q_i}^{q}\right]^{1/q}. \tag{8.66}$$

Inequality (8.66) reduces to an equality if and only if (8.60) and (8.63) are fulfilled. This means that the following two conditions must be satisfied:

(a) all K_i are of the same sign;

(b) relations (8.59) and (8.65) hold true.

Consider relation (8.65). We have

$$\hat{x}_i \equiv L_i^{-1}\|x_i\|_{q_i} \equiv L_i^{-1}\left[\int_a^b |x_i(t)|^{p_i}\,dt\right]^{1/p_i} = \\ = \hat{k}y_i^{q-1} \equiv \hat{k}L_i^{q-1}\|y_i\|_{q_i}^{q-1}. \tag{8.67}$$

Inserting for $|x_i|$ in the integrand its expression from (8.59), we find

$$L_i^{-1}\,|K_i|\,\|y_i\|_{q_i}^{q_i/p_i} = \hat{k}L_i^{q-1}\|y_i\|_{q_i}^{q-1} \tag{8.68}$$

or, solving for $|K_i|$,

$$|K_i| = \hat{k}L_i^{q}\|y_i\|_{q_i}^{q-q_i} \quad \text{for} \quad i=1,\,2,\,\ldots,\,n. \tag{8.69}$$

For this reason, if K_i in (8.59) is chosen so that (8.69) is satisfied, (8.66) is fulfilled automatically. K_i may be either positive or negative; the only point is that they should invariably be of the same sign. This means that (8.69) and conditions (a) and (b) can be replaced by a single condition

$$K_i = kL_i^{q}\|y_i\|_{q_i}^{q-q_i} \tag{8.70}$$

for all $i=1,\,\ldots,\,n$.

Substituting (8.70) in (8.59), we finally obtain

$$x_i(t) = kL_i^{q}\|y_i\|_{q_i}^{q-q_i}\,|y_i(t)|^{q_i-1}\operatorname{sign}y(t) \tag{8.71}$$

for $a \leqslant t \leqslant b$ and all $i=1,\,2,\,\ldots,\,n$.

We now return to our original problem. Consider the functional

$$f_A(k^*)=\int_{t_0}^{t_1} k^*(t_1,\tau)\,u(\tau)\,d\tau. \tag{8.72}$$

Using Hoelder's generalized inequality, we find that

$$|f_A(k^*)|\leqslant \|k^*\|_q\,\|u\|_p; \tag{8.73}$$

and seeing that $f_A(k^*)=\lambda^* l_{ref}(t_1)=1$ we have from (8.47)

$$\|u\|_p \geqslant \frac{|f_A(k^*)|}{\|k^*\|_q}=\frac{1}{\|k^*\|_q}=\|f_A\|. \tag{8.74}$$

Inequality (8.74) is thus a necessary condition for $u(\tau)$ to be an element of the set u_A.

From (8.74) we can now derive a necessary condition for $u(\tau)\in u_A$ to have a minimum norm $\|u\|_p$, specifically:

$$\|u\|_p\frac{1}{\|k^*\|_q}=\|f_A\|, \tag{8.75}$$

which is obtained if the equality is taken in (8.74) and (8.73). The relations obtained from Hoelder's generalized inequality show that the equality in (8.74) is obtained if the components $u_i(\tau)$ are of the form

$$u_i(\tau)=kL_i^q\|k_i^*\|_{q_i}^{q-q_i}\,|k_i^*(t_i,\tau)|^{q_i-1}\operatorname{sign}k_i^*(t_1,\tau). \tag{8.76}$$

Substituting (8.76) in (8.75), we find the constant k,

$$k=\frac{1}{(\|k^*\|_q)^q}. \tag{8.77}$$

Equation (8.75) thus holds true if and only if

$$u_i(\tau)=\frac{L_i^q}{(\|k^*\|_q)^q}\|k_i^*\|_{q_i}^{q-q_i}\,|k_i^*(t_1,\tau)^{q_i-1}|\operatorname{sign}k_i^*(t_1,\tau). \tag{8.78}$$

From the results of Kirillova /17/ we further obtain (putting $q=1$ in (8.75), which corresponds to $p\to\infty$)

$$u_i(t)^*=\frac{L_i}{(\|k^*\|_q)^q}\|k_i^*\|_{q_i}^{q-q_i}\,|k_i^*(t_1,\tau)^{q_i-1}|\operatorname{sign}k_i(t_1,\tau), \tag{8.79}$$

where the asterisk marks those quantities which are determined from λ^*, i.e.,

$$\|k^*\|_1=\min_{\lambda}\sum_{i=1}^{r}L_i\|k_i\|_{q_i} \tag{8.80}$$

on condition $\lambda e_{ref}(t_1)=1$.

To solve the original problem of time optimal control, we have to study the effect of the constraints. As has been shown in the preceding, all the constraints can be summarized by a single condition imposed on the norm of the control vector:

$$\|u\|\leqslant 1. \tag{8.81}$$

Moreover, for $p=\infty$, $q=1$, and

$$\|u^*\| \geqslant \frac{1}{\|k^*\|_1}. \tag{8.82}$$

Hence it follows that for any t_1 the problem is solvable if and only if

$$\|k^*\|_1 \geqslant 1. \tag{8.83}$$

Let $\|k^*\|_1$ be a continuous function of t_1; the minimum time $T_0 = t_1^* - t_0$ is then obtained for the smallest $t_1 = t_1^*$, such that

$$\|k\|_1 = 1, \tag{8.84}$$

and this fact is used in the determination of t_1^*. If we now apply the solution of (8.79) with the minimum norm, we find for $t_1 = t_1^*$

$$\|u^*\| = \frac{1}{\|k^*\|_1}, \tag{8.85}$$

so that the solution given by (8.79) satisfies both the constraints and the final conditions in the shortest time and is thus time-optimal. Using (8.85) we find

$$u_l(t)_{\text{opt}} = L_l \|k_l^*\|_{q_l}^{1-q_l} |k_l^*(t_1^*, t)|^{q_l - 1} \operatorname{sign} k_l^*(t_1^*, t), \tag{8.86}$$

where k_l^* is determined by solving (8.80) for λ^* and substituting the solution in (8.36). From (8.85) we then find the minimum t_1 equal to t_1^*.

We have considered in some detail the theory and the proof of /21/ for the determination of a control function (as a function of time) ensuring a minimum transient time for the problem at hand. Despite the apparent complexity of the method and the introduction of mathematical techniques which are unfamiliar to most engineers, it seems to us that the effort is justified by the simplicity of the final solution. We would like to comment, however, on the practical value of the result. Here the vector of plant inputs (the control vector) is specified as a function of time, and not as a function of the plant outputs. This is equivalent to setting up an open-loop control system, with all the consequences. But there is more to it. The mathematically derived input vector should be implemented in practice, and this requires the introduction of special equipment whose properties have not been allowed for in the mathematical stage. This is a highly significant point in our opinion, since the complete system, including the equipment that implements the control function, is essentially different from the initial system where only the plant properties are relevant. This remark applies to all solution techniques which produce the control vector as a function of time. Further on in this chapter we will consider methods for the derivation of control vectors as a function of the output (controlled) variables.

§ 8.7. APPLICATION OF METHODS OF VARIATIONAL CALCULUS

In this and the following sections we will consider the construction of multivariable control systems whose properties satisfy a certain optimality

test. A frequently used optimality test is minimizing the integral square error of some function of the controlled variables, their derivatives, and plant inputs /30, 66/. The general problem is formulated as follows. For a controlled object of known characteristics, choose the control system, and in particular the controller, so as to satisfy a certain optimality test. This formulation is fully applicable to multivariable control systems. In the latter case, however, the optimality test should correspond to the set of all controlled variables, and not to some individual variable.

This approach was developed by A.M. Letov /30/ for the synthesis of controllers in single-variable systems, and he called his technique the method of analytical controller design. His results are used here insofar as they are applicable to multivariable control systems. *

The mathematical formulation of the variational problem is the following. Let the controlled plant have n controlled variables Y_i and m controllers x_j. Here $m \geqslant n$. A control system is hooked up for each controlled variable. For simplicity we will first assume that each controlled variable is described by a first order equation. Seeing that the controlled variables interact through the plant, we write for the i-th variable

$$Y_i' = +\alpha_{ii} Y_i + \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ik} Y_k + \sum_{j=1}^{m} \beta_{jk} x_k. \tag{8.87}$$

Here α is taken with its algebraic sign. Taking $i = 1, 2, \ldots, n$, we obtain a complete set of differential equations describing the dynamics of a multivariable plant.

The initial conditions for (8.87 are

$$\left.\begin{array}{ll} \text{for } t=0 & \left\{\begin{array}{ll} Y_i = Y_{i0} & (i=1, \ldots, n), \\ X_j = X_{j0} & (j=1, \ldots, m); \end{array}\right. \\ \text{for } t=\infty & \left\{\begin{array}{ll} Y_i = Y_{if} & (i=1, \ldots, n), \\ X_j = X_{jf} & (j=1, \ldots, m). \end{array}\right. \end{array}\right\} \tag{8.88}$$

This set of equations can be written for deviations of the plant inputs and outputs. Taking $Y_i = Y_{i0} + \Delta Y_i$ and considering the deviations only, we should replace Y_i, Y_k, and x_k in (8.87) by ΔY_i, ΔY_k and Δx_k. The final state of a stable system is then described by $\Delta Y_i = 0$, and if the input deviations are reckoned from a new steady-state level, we have

$$\Delta x_i = 0, \quad t = \infty.$$

In the following we will be only concerned with the deviation of the plant inputs from a certain prescribed value, but the equations will be left in the original form (8.87), with Y_i, Y_k, and x_k interpreted as the deviations.

The initial and the final state are then

$$\left.\begin{array}{ll} \text{for } t=0 & \left\{\begin{array}{ll} Y_i = Y_{i0} & (i=1, \ldots, n), \\ X_j = X_{j0} & (j=1, \ldots, m); \end{array}\right. \\ \text{for } t=\infty & \left\{\begin{array}{ll} Y_i = 0 & (i=1, \ldots, n), \\ X_j = 0 & (j=1, \ldots, m). \end{array}\right. \end{array}\right\} \tag{8.89}$$

* Analysis and synthesis of multivariable control systems in a somewhat different form from that presented here were carried out by Ma Fu-wu as part of his post-graduate studies under the direction of the author.

The plant inputs X_j are the controller outputs. The controller structure and parameters are not known at this stage. Our problem is to choose these unknowns so that the control system conforms to a certain optimality test.

The variational problem is thus given the following mathematical formulation. Suppose that the criterion function of the optimality test is the integral

$$Y(X_1, \ldots, X_m) = \int_0^\infty V\,dt, \tag{8.90}$$

where

$$V = \sum_k^n a_k Y_k^2 + \sum_j^m b_j x_j^2. \tag{8.91}$$

The integral (8.90) is a functional defined on a certain class of functions, and its value is the integral square error with constant weights a_k, b_j that the system acquires during a transient $t^* = \infty$. Our aim is to find the analytical expression for the control function

$$\varphi(y_1, \ldots, y_n, x_1, \ldots, x_m) = 0, \tag{8.92}$$

which, in conjunction with the original set of equations (8.87), constitutes a stable system and minimizes the functional (8.90). Meanwhile we are dealing with a linear system, or to use the conventional terminology, equations (8.87) are defined in an open domain. Lagrange's function is

$$H = V + \sum_k \lambda_k \left[Y_k' - \sum_{i \neq 1}^n a_{ik} Y_i - \sum_{j=1}^m \beta_{jk} x_j \right] \tag{8.93}$$

or

$$H = \sum_{i=1}^n a_k Y_k^2 + \sum_j b_j x_j^2 + \sum \lambda_k \left[Y_k' - \sum_{i=1}^n a_{ki} Y_i - \sum_{j=1}^m \beta_{ki} x_j \right], \tag{8.94}$$

where λ_k are Lagrange's multipliers.

We have

$$\left.\begin{aligned} \frac{\partial H}{\partial y_k} &= 2a_k y_k - \sum_i \lambda_i a_{ik}, & \frac{\partial H}{\partial y_k'} &= \lambda_k, \\ \frac{\partial H}{\partial x_j} &= 2b_j x_j - \sum_j \beta_{jk} \lambda_j, & \frac{\partial H}{\partial x_j'} &= 0. \end{aligned}\right\} \tag{8.95}$$

The Euler-Lagrange equations are thus

$$\left.\begin{aligned} \lambda_k' &= 2a_k y_k - \sum_i \lambda_i a_{ik}, \\ 2b_j x_j - \sum_j \beta_{ik} \lambda_j &= 0 \\ (k = 1, \ldots, n;\ j = 1, \ldots, m), & \end{aligned}\right\} \tag{8.96}$$

and these equations, together with (8.87), define the properties of the multivariable control system. Proceeding along the same lines as in /30/, we can find the controller equations for the multivariable system.

Consider the case of a multivariable control system with two controllers and two controlled variables interacting through the plant.

The plant equations are

$$\left.\begin{aligned} y'_1 &= \alpha_{11}y_1 + \alpha_{12}y_2 + \beta_{11}x_1, \\ y'_2 &= \alpha_{21}y_1 + \alpha_{22}y_2 + \beta_{22}x_2. \end{aligned}\right\} \tag{8.97}$$

Unlike the general case, we assume that the controllers do not interact, i.e., $P_{ik}=0$, $i \neq k$. The functional to be minimized is

$$y(x_1, x_2) = \int_0^\infty V\,dt = \int_0^\infty \left[a_1y_1^2 + a_2y_2^2 + b_1x_1^2 + b_2x_2^2\right] dt. \tag{8.98}$$

Lagrange's function for this example is

$$H = a_1y_1^2 + a_2y_2^2 + b_1x_1^2 + b_2x_2^2 + \lambda_1\left[y'_1 - \alpha_{11}y_1 - \alpha_{12}y_2 - \beta_{11}x_1\right] + \\ + \lambda_2\left[y'_2 - \alpha_{21}y_1 - \alpha_{22}y_2 - \beta_{22}x_2\right], \tag{8.99}$$

$$\frac{\partial H}{\partial y_1} = 2a_1y_1 - \lambda_1\alpha_{11} - \lambda_2\alpha_{21}, \tag{8.100}$$

$$\frac{\partial H}{\partial y'_1} = \lambda_1, \tag{8.101}$$

$$\frac{\partial H}{\partial y_2} = 2a_2y_2 - \lambda_1\alpha_{21} - \lambda_2\alpha_{22}, \tag{8.102}$$

$$\frac{\partial H}{\partial y'_2} = \lambda_2, \tag{8.103}$$

$$\frac{\partial H}{\partial x_1} = 2b_1x_1 - \lambda_1\beta_{11}, \tag{8.104}$$

$$\frac{\partial H}{\partial x'_1} = 0, \tag{8.105}$$

$$\frac{\partial H}{\partial x_2} = 2b_2x_2 - \lambda_2\beta_{22}, \tag{8.106}$$

$$\frac{\partial H}{\partial x'_2} = 0. \tag{8.107}$$

From (8.100)—(8.107) and (8.97) we write the Euler—Lagrange equations

$$\left.\begin{aligned} \lambda'_1 &= 2a_1y_1 - \lambda_1\alpha_{11} - \lambda_2\alpha_{21}, \\ \lambda'_2 &= 2a_2y_2 - \lambda_1\alpha_{12} - \lambda_2\alpha_{22}, \\ 2b_1x_1 &- \lambda_1\beta_{11} = 0, \\ 2b_2x_2 &- \lambda_2\beta_{22} = 0, \\ y'_1 &= \alpha_{11}y_1 + \alpha_{12}y_2 + \beta_{11}x_1, \\ y'_2 &= \alpha_{21}y_1 + \alpha_{22}y_2 + \beta_{22}x_2. \end{aligned}\right\} \tag{8.108}$$

The determinant of this system is

$$\Delta = \begin{vmatrix} p-\alpha_{11} & -\alpha_{12} & -\beta_{11} & 0 & 0 & 0 \\ -\alpha_{21} & p-\alpha_{22} & 0 & -\beta_{22} & 0 & 0 \\ 0 & 0 & 0 & 2b_2 & 0 & -\beta_{22} \\ 0 & 0 & 2b_1 & 0 & -\beta_{11} & 0 \\ 0 & -2a_2 & 0 & 0 & \alpha_{12} & p+\alpha_{22} \\ -2a_1 & 0 & 0 & 0 & p+\alpha_{11} & \alpha_{21} \end{vmatrix}. \tag{8.109}$$

After simple manipulations, we obtain the characteristic equation in the form

$$b_1b_2p^4 - \left(\alpha_{11}^2b_1b_2 + \alpha_{22}^2b_1b_2 + 2\alpha_{12}\alpha_{21}b_1b_2 + a_2b_1\beta_{22}^2 + \right. \\ \left. + a_1b_2\beta_{11}^2\right)p^2 + \left(\alpha_{11}^2\alpha_{22}^2b_1b_2 - 2\alpha_{12}\alpha_{21}\alpha_{11}\alpha_{22}b_1b_2 + \right. \\ + a_2\alpha_{11}^2\beta_{22}^2b_1 + a_1\alpha_{22}^2\beta_{11}^2b_2 + \alpha_{12}^2\alpha_{21}^2b_1b_2 + a_1\alpha_{12}^2\beta_{22}^2b_1 + \\ \left. + a_2\alpha_{21}^2\beta_{11}^2b_2 + a_1a_2\beta_{11}^2\beta_{22}^2\right) = 0. \tag{8.110}$$

The parameters of equation (8.110) should be chosen so as to ensure stability of the system. First note that the signs of the coupling coefficients α_{12} and α_{21} depend on the properties of the controlled plant. These coefficients generally have the sign minus, and in what follows we indeed assume negative α_{12} and α_{21}. Strictly speaking, the coupling coefficients may be inherently positive or they can be made positive, as in an electrical system with mutual inductance. These cases are not considered here. In the following we assume that the plant is intrinsically stable (without a controller). Therefore, given the plant equation in the form (8.87), we conclude that α_{ii} are also negative. The weights b_1 and b_2 are positive by definition, and so only the sign of the coefficients β_{11} and β_{22} is unknown. We see from (8.110), however, that this uncertainty is of no consequence at this stage, since β_{11} and β_{22} are squared in the coefficients of (8.110) and their sign is therefore irrelevant. Our preliminary analysis thus shows that the expressions in parentheses in (8.110) are inherently positive. The minus sign in front of one of the coefficients in the free term of (8.110) does not affect this conclusion, since clearly $\alpha_{ii}>\alpha_{ik}$; but even without this condition it is readily seen that in the final account the coefficients of (8.110) are positive.

Let A_0 stand for the coefficient of p^4, A_1 for the coefficient of p^2, and A_2 for the free term in (8.110). Equation (8.110) then takes the form

$$A_0p^4 - A_1p^2 + A_2 = 0.$$

Substituting

$$p^2 = \zeta,$$

we find

$$A_0\zeta^2 - A_1\zeta + A_2 = 0, \tag{8.111}$$

and the solutions are

$$\zeta_{1,2} = \frac{A_1}{2A_0} \pm \sqrt{\frac{A_1^2}{4A_0^2} - \frac{A_2}{A_0}}. \tag{8.112}$$

It is also easily seen that $A_1^2 > 4A_0A_2$, so the roots of (8.112) are positive numbers ζ_1 and ζ_2. The original characteristic equation thus has four roots:

$$\left.\begin{aligned} p_{1,2} &= \pm\sqrt{\zeta_1} \\ \text{and} \qquad p_{3,4} &= \pm\sqrt{\zeta_2}. \end{aligned}\right\} \tag{8.113}$$

We will only use the roots p_1 and p_3, having Re $p<0$. The solution for the inputs and the outputs in this case is

$$y_1 = \sum_{i=1,3} C_i\Delta_1(p_i)e^{p_it}, \tag{8.114}$$

$$y_2 = \sum_{j=1,3} C_j\Delta_2(p_j)e^{p_jt}, \tag{8.115}$$

$$x_1 = \sum_{\rho=1,3} C_\rho\Delta_3(p_\rho)e^{p_\rho t}, \tag{8.116}$$

$$x_2 = \sum_{\nu=1,3} C_\nu\Delta_4(p_\nu)e^{p_\nu t}. \tag{8.117}$$

Here Δ_1, Δ_2, Δ_3, and Δ_4 are respectively the minors of the determinant of (8.110) for the first element of the first row, the second element of the second row, the third element of the third row, and the fourth element of the fourth row.

In order to find x_1, x_2 as a function of y_1, y_2, we should eliminate time between (8.114)—(8.117). The determinant of the controller equations is

$$\begin{vmatrix} \Delta_1(p_1) & \Delta_1(p_3) & y_1 \\ \Delta_2(p_1) & \Delta_2(p_3) & y_2 \\ \Delta_3(p_1) & \Delta_3(p_3) & x_1 \\ \Delta_4(p_1) & \Delta_4(p_3) & x_2 \end{vmatrix} = 0. \tag{8.118}$$

We see from (8.118) that the equations for x_1 and x_2 are linearly dependent. The equation of the controller having x_1 as its output is obtained from

$$\begin{vmatrix} \Delta_1(p_1) & \Delta_1(p_3) & y_1 \\ \Delta_2(p_1) & \Delta_2(p_3) & y_2 \\ \Delta_3(p_1) & \Delta_3(p_3) & x_1 \end{vmatrix} = 0,$$

whence

$$x_1 = \frac{[\Delta_3(p_1)\Delta_2(p_3) - \Delta_2(p_1)\Delta_3(p_3)]\, y_1 - [\Delta_3(p_3)\Delta_1(p_1) - \Delta_1(p_3)\Delta_3(p_1)]\, y_2}{\Delta_1(p_1)\Delta_2(p_3) - \Delta_2(p_1)\Delta_1(p_3)}. \tag{8.119}$$

It is thus clear from this equation that the optimum in the sense of our analysis is ensured if the controller action is influenced by the two controlled variables y_1 and y_2. This confirms our earlier conclusion that interacting control produces an extremum, and a noninteracting control system will therefore give poorer results from the aspect of our optimality test. It is perfectly obvious that the equation of the second controller is also a function of both controlled variables y_1 and y_2.

Another highly significant conclusion from this example is the following. Substituting for $\Delta_i(p_j)$ their values, we obtain a controller with infinitely fast response. This result was also obtained by Letov /30/ in an example of analytical controller design for a single-variable system. The essential point is that a system of this kind can be built only using structures which are stable at infinite gain.

In our discussion of multivariable control systems with infinite gain stability we have shown that interacting control, under certain conditions, has better dynamic response than noninteracting control (see Chapter Four). In the present chapter, in solving the problem of a controller extremizing a certain criterion function, we have established that interacting control is essential for this purpose and that the system must have infinite gain stability. This clearly gives collateral support to our previous assertion that structures stable at infinite gain should be preferred in multivariable control systems.

So far we have been dealing with a controlled plant whose outputs are described by first-order different equations. The controller selection procedure, however, is quite general and can be used with output equations of higher order. In this case, each of the equations of v-th order, say, can be reduced to v first-order equations, and Lagrange's function and the variational equations are then written for each of them separately. The mathematical manipulations are fairly tedious even for the simplest

case of a two-variable system described by first-order equations with inertialess controllers. Although the difficulties are merely technical and can be easily overcome with the aid of modern computers, the results are far from being easy to grasp, especially for the more complex cases. Therefore, following /30/, we will consider some applications of dynamic programming to the design of multivariable control systems.

§ 8.8. BELLMAN'S PRINCIPLE OF OPTIMALITY AND THE FUNCTIONAL EQUATION

In this section we will describe some results due to R. Bellman /7/, which will be used in the following. The brief exposition here cannot be regarded as a substitute for reading Bellman's book /7/, but it will enable the reader to follow the synthesis method proposed for multivariable control systems minimizing or maximizing (according to the particular test used) a certain criterion functional of the system as a whole.

Letov /30/ used Bellman's dynamic programming method for analytical controller design in single-variable systems. In addition to Bellman's results, we will also apply here some of Letov's techniques and conclusions.

1. Multistage allocation process and optimal policy

We start with a certain limited quantity of resources x that can be used to buy equipment of two kinds, A and B. If a certain quantity of resources $0 \leqslant y \leqslant x$ is allocated to purchase equipment A, and the remaining $x-y$ to purchase B, the total return, expressed in terms of labor, say, is

$$R_1(x, y) = g(y) + h(x-y). \tag{8.120}$$

Here $g(y)$ is the return from the allocation y, and $h(x-y)$ the return from $x-y$. The problem is to choose such y in the interval $[0, x]$ that the return $R_1(x, y)$ is maximized. The maximum return is thus

$$R_1(x) = \max_{0 \leqslant y \leqslant x} [g(y) + h(x-y)]. \tag{8.121}$$

If this problem is solvable, we have a single-stage allocation, to use Bellman's terminology /7/.

Consider a multistage process. Suppose that after some time in operation, the equipment is sold, bringing ay money units as the prices of equipment A ($0 \leqslant a < 1$) and $b(x-y)$ units as the price of B. The first stage thus ends with an additional quantity of resources x_1, where

$$x_1 = ay + b(x-y).$$

These resources x_1 are again used to purchase equipment. In this purchase y_1 is allocated to class A and $x_1 - y_1$ to class B; the return is thus

$$g(y_1) + h(x_1 - y_1),$$

where $0 \leqslant y_1 \leqslant x_1$. The total return of the two-stage decision process consisting first in the choice of y and then the choice of y_1 is thus

$$R_2(x, y, y_1) = g(y) + h(x-y) + g(y_1) + h(x_1 - y_1), \tag{8.122}$$

where

and
$$\left.\begin{aligned} x_1 &= ay + b(x-y) \\ 0 \leqslant y \leqslant x, &\quad 0 \leqslant y_1 \leqslant x_1. \end{aligned}\right\} \tag{8.123}$$

The maximum return is attained if y and y_1 are so chosen that $R_2(x_1, y, y_1)$ is maximized under constraints (8.123).

If the buy-and-sell process is repeated n times, we obtain an n-stage allocation process with the total return

$$R_n(x_1, y, y_1, \ldots, y_{n-1}) = g(y) + h(x-y) + g(y_1) + \\ + h(x_1 - y_1) + \ldots + g(y_{n-1}) + h(x_{n-1} - y_{n-1}), \tag{8.124}$$

where

$$x_i = ay_{i-1} + b(x_{i-1} + y_{i-1}) \quad \text{for} \quad i = 1, 2, \ldots, n-1, \tag{8.125}$$

$$y_0 = y \quad \text{and} \quad x_0 = x \ , \tag{8.126}$$

$$0 \leqslant y \leqslant x, \quad 0 \leqslant y_i \leqslant x_i \qquad (i = 1, \ldots, n-1).$$

The maximum total return is attained if the y_i are so chosen that R_n is maximized under constraints (8.125) and (8.126). The fact that the problem is essentially an n-stage decision problem can be applied to simplify the solution and primarily to reduce the number of variables. It is significant that in the k-th stage the problem can be solved if y_{k-1} alone is known. The value of y_{k-1} depends on x_{k-1} and the remaining $N-k$ stages.

Hence, to decide on a solution for the k-th stage it is important to know the resources available at that stage and the number of stages to go; in other words, the problem is as if formulated anew at each stage, with a given number of stages and given quantity of resources. Following /7/, we introduce two new concepts. The sequence of solutions $(y, y_1, \ldots, y_{n-1})$ is called a policy. A policy maximizing the total return according to a certain criterion function is called the optimal policy.

2. Formulation of the problem using functional equations and Bellman's principle of optimality

Let $f_n(x)$ be the total return for an n-stage process with initial resources x and an n-stage optimal policy, $n = 1, 2, \ldots$.

We will derive a recurrence relation for $f_n(x)$ and $f_{n+1}(x)$. Let the initial allocation be y_n, and we consider a $(n+1)$-stage process. If the first-stage return is $g(y) + h(x-y)$, the total return after the $(n+1)$-th stage is $g(y) + h(x-y)$, plus the n-stage return, assuming $x_1 = ay + b(x-y)$ as the resources after the first stage. The essential point is that, irrespective of y, the resources $ay + b(x-y)$ are recovered using an optimal allocation policy in the next n-stages. The total n-stage return will then be $f_n[ay + b(x-y)]$.

Hence, the total return after the $(n+1)$-th stage, with the initial allocation y between 0 and x, is

$$g(y) + h(x-y) + f_n[ay + b(x-y)]. \tag{8.127}$$

Now, y should be chosen so as to maximize (8.127). Since the maximum is numerically equal to the function $f_{n+1}(x)$, we obtain the basic recurrence relation

$$f_{n+1}(x) = \max_{0 \leqslant y \leqslant x} \{g(y) + h(x-y) + f_n[ay + b(x-y)]\}, \qquad (8.128)$$
$$n = 1, 2, \ldots$$

Let us now compare the two formulations of the same problem. In the first formulation, we were expected to choose y_i, $i = 1, \ldots, n-1$, maximizing $R_n(x, y, y_1, \ldots, y_{n-1})$, and in the second formulation we had to select n functions f_n for one y. Bellman /7/ has shown that the second formulation is much more convenient for practical calculations and readily brings out the dependence of the solution on various parameter changes. In the second formulation, the function $f_{n+1}(x)$ can be determined if $f_n(x)$ alone is known. We thus arrive at Bellman's principle of optimality: an optimal policy has the property that whatever the initial state and the initial decisions are, the remaining decisions will constitute an optimal policy with regard to the state resulting from the first decision.

3. The fundamental functional equation

Consider the following problem: maximize the functional

$$S(y) = \int_0^T F(x, y)\, dt \qquad (8.129)$$

with the constraints

and

$$\left.\begin{aligned} \frac{dx}{dt} &= G(x, y), \quad x(0) = C \\ 0 &\leqslant y \leqslant x. \end{aligned}\right\} \qquad (8.130)$$

This is an ordinary variational problem which is solved by the methods of classical variational calculus, with certain conditions imposed on the functions $G(x, y)$ and $F(x, y)$.

Let us consider this problem from the aspect of dynamic programming. R. Bellman has suggested that the variational problem can be treated as a continuous multistage process. In this approach we are not interested in finding y as a function of t for $0 \leqslant t \leqslant T$ but rather $Y(0)$ as a function of the initial state $X(0) = C$ and the time interval T; in other words, we are looking for a functional equation

$$f[X(0), T] = f(C, T) = \max_y Y(y). \qquad (8.131)$$

Let $F(x, y)$ and $G(x, y)$ respectively ensure the existence of a maximum and the continuity of $f(C, T)$ as a function of C and T. It is moreover assumed that f has continuous partial derivatives with respect to C and T in any bounded region $C \geqslant 0$ and $T \geqslant 0$.

Along the extremal y we have

$$f(C, \rho + T) = \int_0^{\rho} F(x, y)\, dt + \int_{\rho}^{\rho+T} F(x, y)\, dt. \qquad (8.132)$$

For $t=\rho$, x is equal to $x(\rho)$, which is found from the equation

$$\frac{dx}{dt}=G(x,\ y).$$

Let $x(\rho)=x(C)$.

According to the principle of optimality we obtain along the extremal

$$\int_{\rho}^{\rho+T} F(x,\ y)\,dt=f[C(\rho),\ T]. \tag{8.133}$$

The integral (8.129) is thus replaced by the equation

$$f(C,\ \rho+T)=\int_{0}^{\rho} F(x,\ y)\,dt+f[C(\rho),\ T]. \tag{8.134}$$

Now y is chosen so as to maximize (8.134). Hence

$$f[C,\ \rho+T]=\max_{0\leqslant y\leqslant x}^{y[0,\delta]}\left\{\int_{0}^{\rho} F(x,\ y)\,dt+f[C(\rho),\ T]\right\}. \tag{8.135}$$

Let $F(x,\ y)$ be continuous in x and in y, and have continuous partial derivatives with respect to $C(f_C)$ and $T(f_T)$; if moreover y is a continuous function of t, then for small ρ we may write

$$f(C,\ \rho+T)=f(C,\ T)+\rho\frac{\partial f}{\partial T}+o(\rho), \tag{8.136}$$

$$C(\rho)=C+\rho G(C,\ \nu)+o(\rho), \tag{8.137}$$

$$f(C(\rho),\ T)=f(C,\ T)+\rho G(C,\ \nu)\frac{\partial f}{\partial C}+o(\rho) \tag{8.138}$$

and

$$\int_{0}^{\rho} F(x,\ y)\,dt=\rho F(C,\ \nu)+o(\rho). \tag{8.139}$$

Here $\nu=y(0)=\nu(C,\ T)$. In the limit, as $\rho\to 0$, we find

$$f_T=\max_{0\leqslant\nu\leqslant C}\left[F(C,\ \nu)+G(C,\ \nu)\frac{\partial f}{\partial C}\right]. \tag{8.140}$$

In our earlier notation, taking as the criterion function the integral $\int_{0}^{\infty} F(x,\ y)\,dt$ with the constraints

$$\frac{dx}{dt}=G(x,\ y),$$

we write Bellman's functional equation in the form

$$\min_{y}\left[F(x,\ y)+G(x,\ y)\frac{\partial f}{\partial x}\right]=f_T. \tag{8.141}$$

The material of this section is sufficient for further discussion.

§ 8.9. APPLICATION OF DYNAMIC PROGRAMMING TO THE SYNTHESIS OF MULTIVARIABLE CONTROL SYSTEMS

Consider a controlled object with n outputs (controlled variables) interacting through the plant and m controlling inputs; here $m>n$. The controlled variables are again denoted by y_i and the controlling inputs by x_j; here $i=1, \ldots, n$, and $j=1, \ldots, m$.

The equation of motion of the system can be written in matrix form as

$$\frac{dY}{dt} = AY + BX, \tag{8.142}$$

where A is the matrix of plant parameters and coupling coefficients, B the matrix of controller coefficients, Y and X are column vectors. For $t=0$, we take $Y=Y(0)$.

The criterion function is the integral

$$I(X) = \int_0^\infty V\,dt. \tag{8.143}$$

The problem is to find a control function X as a function of the plant states which minimizes the functional (8.143) /30/.

In (8.134) it is implied that each output is described by a first-order differential equation. The perturbation equation can be easily written for the general case and then reduced to the form (8.142). The problem then can be stated as follows. Suppose that the i-th output is described by an equation of V_i-th order. Reducing the V_i-th order equation to V_i first-order equations, we find

$$\frac{dY_i^{V_i}}{dt} = \sum_{k=1}^{V_i} a_v Y^k + \sum_{\rho=1}^{n} a_{i\rho} Y_\rho - \sum_{j=1}^{m} \beta_{ij} X_j, \tag{8.144}$$

$$Y^k = -\frac{dY^{k-1}}{dt} \qquad (i=1, \ldots, n;\ k=1, \ldots, V_{i-1}).$$

For simplicity we will henceforth assume that each output is described by a first-order equation. In principle this restriction is of no consequence, since in the more general case the equation is reduced to the form (8.142) and the synthesis methods derived in the following are directly applicable.

In expanded form (8.142) is written as

$$\frac{dY_i}{dt} = \sum_{k=1}^{n} a_{ik} Y_k + \sum_{j=1}^{m} \beta_{ij} X_j \qquad (i=1, \ldots, n), \tag{8.145}$$

$$Y_i(t)=0, \qquad Y_i(0)=Y_{i0}.$$

To simplify the notation, (8.145) can be written as

$$\frac{dY_i}{dt} = G_i(Y_1, \ldots, Y_n, X_1, \ldots, X_m) = G_i(Y, X) \tag{8.146}$$

$$(i=1, \ldots, n).$$

The criterion function is the integral (8.143) with

$$V = \sum_{i=1}^{n} a_i Y_i^2 + \sum_{j=1}^{m} b_j X_j^2. \tag{8.147}$$

Here a_i and b_j are known nonnegative weight coefficients, whose values are chosen according to the desired sharpness of the minimum in each controlled variable.

Solution for an open domain

In an open domain $G(Y, X)$ the function is minimized over a class of functions where X_i and Y_i are continuous and continuously differentiable.

Following the procedure of /30/ for single-variable systems, we solve the problem for multivariable control systems using Bellman's dynamic programming method.

Let the functions $X=X(x_1, \ldots, x_m)$ minimize the functional (8.143). It is clear from the preceding discussion of Bellman's results that the minimum of $I(X)$ is a certain function $\psi(Y_0)$ of the initial state of the system. We may therefore write

$$\min_X I(X) = \psi(Y_0). \tag{8.148}$$

Bellman's conditions for our case take the following form. For a positive ρ we may write

$$I(X) = \int_0^\infty V\,dt = \int_0^\rho V\,dt + \int_\rho^\infty V\,dt. \tag{8.149}$$

By Bellman's principle of optimality it is clear that, irrespective of the choice of the function $X[0, \rho]$ over $[0, \rho]$, the function $X[\rho, \infty]$ over $[\rho, \infty]$ minimizing the functional $\int_\rho^\infty V\,dt$ can be chosen as if minimizing the functional $\int_0^\infty V\,dt$, with the difference that Y_ρ takes on the role of the initial state at the time $t=\rho$. Hence,

$$\min_{X[\rho, \infty]} \int_\rho^\infty V\,dt = \psi(Y_\rho). \tag{8.150}$$

Therefore, by (8.149) and (8.150),

$$\psi(Y_0) = \min_{X[0, \rho]} \left[\int_0^\rho V\,dt + \psi(Y_\rho) \right]. \tag{8.151}$$

Let ρ be sufficiently small. Then, if the function ψ is differentiable with respect to Y_i for $t \in [0, \rho]$, we have

$$\psi(Y_\rho) = \psi[Y_0 + G\rho] = \psi(Y_0) + \sum_i G_i \rho \left(\frac{\partial \psi}{\partial y_i} \right)_{t=\zeta} + o(\rho). \tag{8.152}$$

Here

$$Y_0 + G\rho = (Y_{01} + G_1\rho, Y_{02} + G_2\rho, \ldots, Y_{0n} + G_n\rho) \quad (0 < \zeta < \rho).$$

As for the residual term $o(\rho)$, we have for $\rho \to 0$

$$\lim_{\rho \to 0} \frac{o(\rho)}{\rho} = 0.$$

Equation (8.152) can be written as

$$\psi(Y_0) = \min_{X[0,\,\rho]} \left[V\rho + \psi(Y_0) + \sum_{i}^{n} G_i \rho \left(\frac{\partial \psi}{\partial Y_i} \right)_{t=\zeta} + o(\rho) \right]. \tag{8.153}$$

For $\rho \to 0$ the interval $[0, \rho]$ contracts to the single point 0, and the choice of the function $X[0, \rho]$ over this interval reduces to the choice of $X(0)$.

Passing to the limit as $\rho \to 0$, we obtain the equation in explicit form

$$\min_{X_0} \left[\sum_{i=1}^{n} a_i y_i^2 + \sum_{j=1}^{m} b_j x_j^2 + \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} y_k + \sum_{j=1}^{m} \beta_{ij} x_j \right) \frac{\partial \psi}{\partial y_i} \right]. \tag{8.154}$$

In order for (8.154) to give a minimum in x_i, the derivatives of (8.154) with respect to $x_1, \ldots, x_m$ should vanish. For the solution of our problem we thus have $m+1$ equations, that is

$$\sum_{i=1}^{n} a_i y_i^2 + \sum_{j=1}^{m} b_j x_j^2 + \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} y_k + \sum_{j=1}^{m} \beta_{ij} x_j \right) \frac{\partial \psi}{\partial y_j} = 0, \tag{8.155}$$

$$2 b_j x_j + \sum_{i=1}^{n} \beta_{ij} \frac{\partial \psi}{\partial y_i} = 0 \qquad (j = 1, \ldots, m). \tag{8.156}$$

From (8.156) we have

$$x_j = -\frac{1}{2b_j} \sum_{i=1}^{n} \beta_{ij} \frac{\partial \psi}{\partial y_i} \qquad (j = 1, \ldots, m). \tag{8.157}$$

Substituting x_j from (8.157) into (8.155), we find

$$\sum_{i=1}^{n} a_i y_i^2 + \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{ik} y_k \frac{\partial \psi}{\partial y_i} = \sum_{j=1}^{m} b_j \left(\frac{1}{2b_j} \sum_{i=1}^{n} \beta_{ij} \frac{\partial \psi}{\partial x_i} \right)^2 + \\ + \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} \frac{1}{2b_j} \sum_{i=1}^{n} \beta_{ij} \frac{\partial \psi}{\partial x_i}. \tag{8.158}$$

We have ended up with a linear partial differential equation. Its solution gives the sought relationship between the controller outputs and the controlled variables, i.e., the controller inputs.

Example

Consider a controlled plant shown in Figure 8.6. The controlled variables Y_1 and Y_2 interact through the plant, as is shown by the direct coupling in the block diagram.

The gain parameters and the time constants are K_{11}, K_{22}, T_{11}, T_{22} for each self-variable and K_{12}, K_{21}, T_{12}, T_{21} for the coupling elements. In what follows all the variable symbols stand for the deviations of the corresponding variables from steady-state values. The plant equation in Laplace transforms is

$$Y_1(p) = \frac{K_{11}}{1+T_{11}p}\mu_1(p) + \frac{K_{12}}{1+T_{12}p}\mu_2(p), \tag{8.159}$$

$$Y_2(p) = \frac{K_{21}}{1+T_{21}p}\mu_1(p) + \frac{K_{22}}{1+T_{22}p}\mu_2(p), \tag{8.160}$$

where μ_1 is the controller input for the first variable, μ_2 ditto for the second variable.

The controller inputs and outputs are related by

$$\mu_1(p) = K_1 x_1(p) \tag{8.161}$$

and

$$\mu_2(p) = K_2 x_2(p). \tag{8.162}$$

Eliminating μ_1, μ_2 between (8.159) and (8.162) and changing back to the originals, we write

$$T_{11}T_{12}\frac{d^2y_1}{dt^2} + (T_{11}+T_{12})\frac{dy_1}{dt} + y_1 = K_1K_{11}T_{12}\frac{dx_1}{dt} + \\ + K_2K_{12}T_{11}\frac{dx_2}{dt} + K_{11}K_1x_1 + K_2K_{12}x_2, \tag{8.163}$$

$$T_{22}T_{21}\frac{d^2y_2}{dt^2} + (T_{22}+T_{21})\frac{dy_2}{dt} + y_2 = K_1K_{21}T_{22}\frac{dx_1}{dt} + \\ + K_2K_{22}T_{21}\frac{dx_2}{dt} + K_1K_{21}x_1 + K_2K_{22}x_2. \tag{8.164}$$

To simplify (8.163), (8.164), we substitute

$$\frac{dy_1}{dt} = y_{12}, \quad \frac{dy_2}{dt} = y_{22}, \quad \frac{dx_1}{dt} = x_{12}, \quad \frac{dx_2}{dt} = x_{22}. \tag{8.165}$$

In this notation, equations (8.163), (8.164) are written as

$$\left.\begin{aligned} \dot{y}_{12} &= -a_{11}y_{12} - a_{12}y_1 + a_{13}x_{12} + a_{14}x_{22} + a_{15}x_1 + a_{16}x_2, \\ \dot{y}_1 &= y_{12}, \\ \dot{x}_1 &= x_{12}, \\ \dot{y}_2 &= y_{22}, \\ \dot{x}_2 &= x_{22}, \\ \dot{y}_{22} &= -a_{22}y_{22} - a_{21}y_2 + a_{23}x_{12} + a_{24}x_{22} + a_{25}x_1 + a_{26}x_2, \end{aligned}\right\} \tag{8.166}$$

where

$$a_{11} = \frac{T_{11}+T_{12}}{T_{11}T_{12}}, \quad a_{12} = \frac{1}{T_{11}T_{12}}, \quad a_{13} = \frac{K_1K_{11}T_{12}}{T_{11}T_{12}},$$
$$a_{14} = \frac{K_2K_{12}T_{11}}{T_{11}T_{12}}, \quad a_{15} = \frac{K_{11}K_1}{T_{11}T_{12}}, \quad a_{16} = \frac{K_{22}K_2}{T_{11}T_{12}},$$
$$a_{22} = \frac{T_{22}+T_{21}}{T_{22}T_{21}}, \quad a_{21} = \frac{1}{T_{22}T_{21}}, \quad a_{23} = \frac{K_1K_{21}T_{22}}{T_{22}T_{21}},$$
$$a_{24} = \frac{K_2K_{22}T_{21}}{T_{22}T_{21}}, \quad a_{25} = \frac{K_1K_{21}}{T_{22}T_{21}}, \quad a_{26} = \frac{K_2K_{22}}{T_{22}T_{21}}.$$

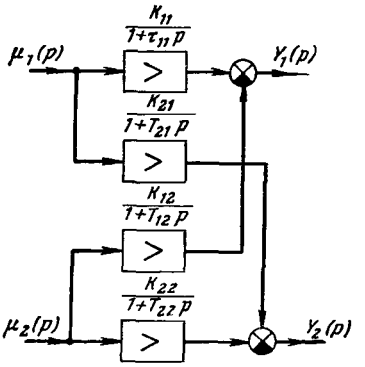

FIGURE 8.6. A multivariable control system.

The problem is stated as follows. Find the control vector x_1, x_{12}, x_2, x_{22} as a function of the system outputs y_1, y_{12}, y_2, y_{22} that minimizes the functional

$$1 = \int_0^\infty [\alpha_{11}y_1^2 + \alpha_{12}(y_{12}^2) + \alpha_{22}y_2^2 + \alpha_{21}(y_{22})^2 + \\ + \beta_{11}x_1^2 + \beta_{12}(x_{12})^2 + \beta_{22}x_2^2 + \beta_{21}(x_{22})^2]\,dt \qquad (8.167)$$

in an open domain $N(y_1, y_{11}, y_2, y_{22}, x_1, x_{12}, x_2, x_{22})$. Here α_{ik} and β_{ik} are known positive numbers.

The corresponding functional equation is

$$\alpha_{11}y_1^2 + \alpha_{12}y_{12}^2 + \alpha_{22}y_2^2 + \alpha_{21}y_{22}^2 + \beta_{11}x_1^2 + \beta_{12}x_{12}^2 + \beta_{22}x_2^2 + \\ + \beta_{21}x_{22}^2 + (-a_{11}y_{12} - a_{12}y_1 + a_{13}x_{12} + a_{14}x_{22} + a_{15}x_1 + a_{16}x_2)\frac{\partial\psi}{\partial y_{12}} + \\ + (-a_{22}y_{22} - a_{21}y_2 + a_{23}x_{12} + a_{24}x_{22} + a_{25}x_1 + a_{26}x_2)\frac{\partial\psi}{\partial y_{22}} + \\ + y_{12}\frac{\partial\psi}{\partial y_1} + y_{22}\frac{\partial\psi}{\partial y_2} = 0. \qquad (8.168)$$

Additional equations are obtained by setting the derivatives of (8.168) with respect to x_1, x_{12}, x_2, x_{22} equal to zero. We have

w. r. t. x_1

$$2\beta_{11}x_1 + a_{15}\frac{\partial\psi}{\partial y_{12}} + a_{25}\frac{\partial\psi}{\partial y_{22}} = 0, \qquad (8.169)$$

w. r. t. x_{12}

$$2\beta_{12}x_{12} + a_{13}\frac{\partial\psi}{\partial y_{12}} + a_{23}\frac{\partial\psi}{\partial y_{22}} = 0, \qquad (8.170)$$

w. r. t. x_2

$$2\beta_{22}x_2 + a_{16}\frac{\partial\psi}{\partial y_{12}} + a_{26}\frac{\partial\psi}{\partial y_{22}} = 0, \qquad (8.171)$$

w. r. t. x_{22}

$$2\beta_{21}x_{22} + a_{14}\frac{\partial\psi}{\partial y_{12}} + a_{24}\frac{\partial\psi}{\partial y_{22}} = 0. \qquad (8.172)$$

Hence,

$$x_1 = -\frac{1}{2\beta_{11}}\left[a_{15}\frac{\partial\psi}{\partial y_{12}} + a_{25}\frac{\partial\psi}{\partial y_{22}}\right], \qquad (8.173)$$

$$x_{12} = -\frac{1}{2\beta_{12}}\left[a_{13}\frac{\partial\psi}{\partial y_{12}} + a_{23}\frac{\partial\psi}{\partial y_{22}}\right], \qquad (8.174)$$

$$x_2 = -\frac{1}{2\beta_{22}}\left[a_{16}\frac{\partial\psi}{\partial y_{12}} + a_{26}\frac{\partial\psi}{\partial y_{22}}\right], \qquad (8.175)$$

$$x_{22} = -\frac{1}{2\beta_{21}}\left[a_{14}\frac{\partial\psi}{\partial y_{12}} + a_{24}\frac{\partial\psi}{\partial y_{22}}\right]. \qquad (8.176)$$

Substituting x_1, x_2, x_{12}, and x_{22} from (8.173)—(8.176) into (8.168), we find

$$\begin{aligned}
&\alpha_{11}y_1^2+\alpha_{12}y_{12}^2+\alpha_{22}y_2^2+\alpha_{21}y_{22}^2+\beta_{11}\left[-\frac{1}{2\beta_{11}}\left(a_{15}\frac{\partial\psi}{\partial y_{12}}+a_{25}\frac{\partial\psi}{\partial y_{22}}\right)\right]^2+\\
&+\beta_{12}\left[-\frac{1}{2\beta_{12}}\left(a_{13}\frac{\partial\psi}{\partial y_{12}}+a_{23}\frac{\partial\psi}{\partial y_{22}}\right)\right]^2+\\
&+\beta_{22}\left[-\frac{1}{2\beta_{22}}\left(a_{16}\frac{\partial\psi}{\partial y_{12}}+a_{26}\frac{\partial\psi}{\partial y_{22}}\right)\right]^2+\\
&+\beta_{21}\left[-\frac{1}{2\beta_{21}}\left(a_{14}\frac{\partial\psi}{\partial y_{12}}+a_{24}\frac{\partial\psi}{\partial y_{22}}\right)\right]^2+\\
&+\left[-a_{11}y_{12}-a_{12}y_1-\frac{a_{13}}{2\beta_{12}}\left(a_{13}\frac{\partial\psi}{\partial y_{12}}+a_{23}\frac{\partial\psi}{\partial y_{22}}\right)-\right.\\
&-\frac{a_{14}}{2\beta_{21}}\left(a_{14}\frac{\partial\psi}{\partial y_{12}}+a_{24}\frac{\partial\psi}{\partial y_{22}}\right)-\frac{a_{15}}{2\beta_{11}}\left(a_{15}\frac{\partial\psi}{\partial y_{12}}+a_{25}\frac{\partial\psi}{\partial y_{22}}\right)-\\
&\left.-\frac{a_{16}}{2\beta_{22}}\left(a_{16}\frac{\partial\psi}{\partial y_{12}}+a_{26}\frac{\partial\psi}{\partial y_{22}}\right)\right]+a_{22}y_{22}-a_{21}y_2-\\
&-\frac{a_{23}}{2\beta_{12}}\left(a_{13}\frac{\partial\psi}{\partial y_{12}}+a_{23}\frac{\partial\psi}{\partial y_{22}}\right)-\frac{a_{24}}{2\beta_{21}}\left(a_{14}\frac{\partial\psi}{\partial y_{12}}+a_{24}\frac{\partial\psi}{\partial y_{22}}\right)-\\
&-\frac{a_{25}}{2\beta_{11}}\left(a_{15}\frac{\partial\psi}{\partial y_{12}}+a_{25}\frac{\partial\psi}{\partial y_{22}}\right)-\frac{a_{26}}{2\beta_{22}}\left(a_{16}\frac{\partial\psi}{\partial y_{12}}+a_{26}\frac{\partial\psi}{\partial y_{22}}\right)+\\
&+y_{12}\frac{\partial\psi}{\partial y_1}+y_{22}\frac{\partial\psi}{\partial y_2}=0. \qquad (8.177)
\end{aligned}$$

We have obtained a nonlinear partial differential equation. Its solution is sought in the form

$$\begin{aligned}
\psi=&C_{11}y_1^2+C_{22}y_{12}^2+C_{33}y_2^2+C_{44}y_{22}^2+\\
&+C_{12}y_1y_{12}+C_{13}y_1y_2+C_{14}y_1y_{22}+C_{23}y_{12}y_2+\\
&+C_{24}y_{12}y_{22}+C_{34}y_2y_{22}. \qquad (8.178)
\end{aligned}$$

Here C_{ik} are unknown coefficients.

It thus remains to find all the partial derivatives $\frac{\partial\psi}{\partial y_{ik}}$ from (8.178) and insert the results in (8.169), (8.170), (8.171), and (8.172). This will enable us to find the coefficients C_{ik}. From the entire set of solutions we should select those C_{ik} which ensure stability of the multivariable control system.

BIBLIOGRAPHY

1. AIZERMAN, M. A. and F. R. GANTMAKHER. Absolyutnaya ustoichivost' reguliruemykh sistem (Absolute Stability of Controlled Systems). — Izdatel'stvo AN SSSR. 1963.
2. ANDRONOV, A. A., A. L. VITT, and S. E. KHAIKIN. Teoriya kolebanii (Theory of Oscillations). — Gostekhizdat. 1956.
3. AKHMETGALEEV, I. I. K teorii dvukhkanal'nykh sistem avtomaticheskogo regulirovaniya (The Theory of Two-Channel Control Systems). — Trudy Kazanskogo Aviatsionnogo Instituta. 1960.
4. BARANCHUK, E. I. Issledovanie tochnosti svyazannykh reguliruemykh i sledyashchikh sistem, nakhodyashchikhsya pod vozdeistviem statsionarnykh sluchaev vozdeistvii (Accuracy of Interacting Control and Servosystems under Steady-State Conditions). — Nauchnye Doklady Vysshei Shkoly, Elektronika i Avtomatika, No. 1. 1958.
5. BEIRAKH, Z. Ya. Dinamika regulirovaniya barabannogo parovogo kotla (Boiler Control Dynamics). — Avtomatika i Telemekhanika, No. 2. 1939.
6. BEIRAKH, Z. Ya. Avtomaticheskoe regulirovanie kotel'nykh ustanovok (Automatic Control of Boiler Installations). — In: "Avtomatizatsiya elektrostantsii i energoustanovok". Oborongiz. 1948.
7. BELLMAN, R. Dynamic Programming. — Princeton Univ. Press. 1957.
8. BERMAN, M. A., S. V. EMEL'YANOV, and V. A. TARAN. Dvizhenie sistemy s peremennoi strukturoi v skol'zyashchem rezhime (The Motion of a Sliding-Action Variable-Structure System). — Avtomatika i Telemekhanika, No. 1. 1965.
9. VENIKOV, V. A. and L. A. ZHUKOV. Perekhodnye protsessy v elektricheskikh sistemakh (Transients in Electric Systems). — Gosenergoizdat. 1953.
10. VOZNESENSKII, I. N. K voprosu o vybore skhemy regulirovaniya teplofikatsionnykh turbin (Choice of Control System for Thermal Turbines). — In: "Za sovetskoe energooborudovanie", No. 6. Gosenergoizdat. 1934.
11. VOZNESENSKII, I. N. O regulirovanii mashin s bol'shim chislom reguliruemykh parametrov (Control of Machines with Numerous Controlled Parameters). — Avtomatika i Telemekhanika, No. 4—5. 1938.
12. JEFFREY, A. Stability of Interacting Control Systems.— Proc. First IFAC Congr., Vol. 1. 1960.
13. DRUZHININ, N. P. Control and Coupling Equations for an Electric Drive and Continuous Rolling Mill. — Proc. First IFAC Congress, Vol. 2. 1960.

14. IVAKHNENKO, A. G. Kombinirovannoe regulirovanie kak obshchii sluchai regulirovaniya sostoyaniya i kolichestva (Combined Control as a General Case of State and Quantity Control). — In: "Teoriya invariantnosti i ee primenenie v avtomaticheskikh ustroistvakh". Izdatel'stvo AN SSSR. 1959.
15. INOSOV, V. L. Regulirovanie neskol'kikh parametrov (Multiparameter Control). — Sbornik nauchno tekhnicheskikh statei, No. 1. 1948.
16. KIRILLOV, I. I. Regulirovanie parovykh i gazovykh turbin (Control of Steam and Gas Turbines). — Gosenergoizdat. 1952.
17. KIRILLOVA, F. M. O predel'nom perekhode pri reshenii odnoi zadachi optimal'nogo regulirovaniya (On Passing to the Limit in the Solution of a Problem of Optimal Control). — Prikladnaya Matematika, **24**, No. 2. 1961.
18. KORNILOV, Yu. G. Avtonomnoe regulirovanie kak ekstremal'naya zadacha (Noninteracting Control as an Extremum Problem). — Izvestiya AN SSSR, Dept. Tech. Sci., No. 4. 1954.
19. KORNILOV, Yu. G. Avtonomnoe regulirovanie pri parallel'noi rabote kotel'nykh agregatov (Noninteracting Control for Parallel-Operation Boiler Complexes). — In: "Teoriya invariantnosti i ee primenenie v avtomaticheskikh ustroistvakh", AN UkrSSR. 1950.
20. KOSTYUK, O. M. Korektsiya po temperaturi petli histerezisu ta neliniinostiv v systemakh avtomatichnogo regulyrivoniya za dopomogoyu zvygnykh zvyazkiv (Temperature Compensation of the Hysteresis Loop and Nonlinearities in Control Systems with Coupling). — Avtomatika, No. 3. 1960.
21. KRANC, G. M. and P. E. SARACHIK. On Optimal Control of Systems with Multinorm Constraints. — Proc. Second IFAC Congr. 1963.
22. KRASOVSKII, A. A. K teorii dvukhkanal'nykh sledyashchikh sistem s releinym elementom v tsepi peremennogo toka (Theory of Two-Channel Servosystems with an A. C. Relay Element). — Avtomatika i Telemekhanika, 21, No. 9. 1960.
23. KRASOVSKII, A. A. O dvukhkanal'nykh sistemakh avtomaticheskogo regulirovaniya s antisimmetrichnymi svyazyami (Two-Channel Control Systems with Anti-symmetric Coupling). — Avtomatika i Telemekhanika, 18, No. 2. 1957.
24. KRASOVSKII, A. A. O protsessakh avtomaticheskogo regulirovaniya v odnotipnykh svyazannykh sistemakh (Control of Homogeneous Coupled Systems). 1955.
25. KRASOVSKII, N. N. K teorii optimal'nogo upravleniya (Optimal Control Theory).— Avtomatika i Telemekhanika, 18. Nov. 1957.
26. KULEBAKIN, V. S. Theory of Invariance of Automatic Control Systems. — Proc. First IFAC Congr., Vol. 1. 1960.
27. KULEBAKIN, V. S. Vysokokachestvennye invariantnye sistemy regulirovaniya (Invariant Quality Control Systems). — In: " Teoriya invariantnosti i ee primenenie v avtomaticheskikh ustroistvakh". Izdatel'stvo AN UkrSSR. 1959.
28. KULIKOWSKI, R. K. Optimal Processes and Synthesis of Optimal Systems with Linear and Nonlinear Fixed Components. — Proc. First IFAC Congr., Vol. 2. 1960.

29. KUKHTENKO, A. I. Zadachi invariantnosti do ε dlya sistem regulirovaniya po otkloneniyu (Problems of ε-Invariance for Deviation Control Systems). — In: "Teoriya invariantnosti i ee primenenie v avtomaticheskikh ustroistvakh", Izdatel'stvo AN UkrSSR. 1959.
30. LETOV, A. M. Analiticheskoe konstruirovanie regulyatorov (Analytical Controller Design). — Avtomatika i Telemekhanika, Nos. 4, 5, 6. 1960; No. 4. 1961.
31. LUZIN, N. N. and P. N. KUZNETSOV. Ob absolyutnoi invariantnosti i invariantnosti do ε v teorii differentsial'nykh uravnenii (Absolute Invariance and ε-Invariance in the Theory of Differential Equations). — Doklady AN SSSR, No. 4. 1946.
32. LUZIN, N. N. K izucheniyu matrichnoi teorii differentsial'nykh uravnenii (Studies in the Matrix Theory of Differential Equations). — Avtomatika i Telemekhanika, No. 5. 1940.
33. LYUSTERNIK, L. A. and V. I. SOBOLEV. Elementy funktsional'nogo analiza (Elements of Functional Analysis). — Gostekhizdat. 1956.
34. MEEROV, M. V. Issledovanie i postroenie mnogokonturnykh sistem so mnogimi reguliruemymi velichinami, obladayushchimi vysokoi stepen'yu ustanovivsheisya tochnosti (Analysis and Synthesis of Multivariable Multiloop Control Systems with High Steady-State Accuracy). — In A. A. Andronov's Memorial Volume, Izdatel'stvo AN SSSR. 1955.
35. MEEROV, M. V. K sintezu sistem mnogosvyaznogo regulirovaniya pri nalichii elementov s zapazdyvaniem (Synthesis of Multivariable Control Systems with Lagged Elements). — Avtomatika i Telemekhanika, No. 12. 1957.
36. MEEROV, M. V. Nekotorye printsipy postroeniya mnogokonturnykh sistem so mnogimi reguliruemymi velichinami, obladayushchimi bol'shoi stepen'yu ustanovivsheisya tochnosti (Some Principles of the Design of Multivariable Multiloop Control Systems with High Steady-State Accuracy). — Trudy 2-go soveshchaniya po teorii avtomaticheskogo regulirovaniya, Vol. 1, pp. 479—500. 1955.
37. MEEROV, M. V. Ob avtonomnosti mnogosvyaznykh sistem, ustoichivykh pri neogranichennom uvelichenii ustanovivsheisya tochnosti (Noninteraction of Multivariable Control Systems which are Stable at Infinite Steady-State Gain). — Avtomatika i Telemekhanika, No. 5. 1956.
38. MEEROV, M. V. Sistemy mnogosvyaznogo regulirovaniya (Multivariable Control Systems). — *Izvestiya AN SSSR, Dept. Tech. Sci., Power Engng. and Autom.*, No. 6. 1961.
39. MEEROV, M. V. Sintez struktur sistem avtomaticheskogo regulirovaniya vysokoi tochnosti (Synthesis of Structures for High-Accuracy Control Systems). — Fizmatgiz. 1959.
40. MEEROV, M. V. and E. S. SALIMZHANOV. Nekotorye voprosy teorii upravleniya neftyanymi skvazhinami (Some Aspects of the Theory of Oil Well Control). — *Tekhnicheskaya Kibernetika*, No. 3. 1963.
41. MEEROV, M. V. Synthesis of Fixed Structure Systems Equivalent to Self-Adaptive Systems. — Second Intern. IFAC Congr., Basel. 1963.

42. MEEROV, M. V. O vozmozhnosti podavleniya pomekh v odnom klasse sistem avtomaticheskogo regulirovaniya (Noise Suppression in a Certain Class of Control Systems). — Avtomatika i Telemekhanika, No. 5. 1962.
43. MEEROV, M. V. O strukturnoi pomekhoustoichivosti odnogo klassa sistem regulirovaniya (Noise-Proof Structures in a Certain Class of Control Systems). — Doklady AN SSSR, No. 4. 1961.
44. MEEROV, M. V. Some Structural Features of Multivariable Control Systems. — Proc. First IFAC Congr. 1960
45. MITCHELL, D. C. and C. R. WEBB. A Study of Interaction in Multiloop Control Systems. — Proc. First IFAC Congress., Vol. 1. 1960.
46. MIKHNEVICH, G. V. K analizu sobstvennykh kolebanii slozhnoi parosistemy (Analysis of Natural Oscillations in a Complex Steam Aggregate). — Izvestiya AN SSSR, Dept. Tech. Sci., No. 6. 1962.
47. MIKHNEVICH, G. V. and G. F. KOZLOVSKII. Avtomaticheskoi regulirovanie vozbuzhdeniya sinkhronnykh generatorov v energeticheskoi sisteme s neskol'kimi generatornymi stantsiyami (Automatic Control of Synchronous Machine Excitation in a Power System with Several Generator Stations). — Elektroenergetika, No. 6. 1962.
48. MOROZOVSKII, V. G. O vybore ratsional'noi struktury kompensiruyushchikh perekrestnykh svyazei v mnogokanal'nykh SAR (Selecting Rational Structure of Compensating Cross-Coupling in Multichannel Control Systems). — Izvestiya AN SSSR, power engng. and autom., No. 2. 1961.
49. MOROZOVSKII, V. G. K teorii odnotipnykh svyazannykh sistem avtomaticheskogo regulirovaniya s simmetrichnymi perekrestnymi svyazyami (Theory of Homogeneous Coupled Control Systems with Symmetric Cross-Coupling). — Avtomatika i Telemekhanika, No. 3. 1961.
50. MOROZOVSKII, V. G. Ustoichivost' parallel'noi raboty odnotipnykh sinkhronnykh generatorov (Stability of Identical Synchronous Generator Operating in Parallel). — Izvestiya AN SSSR, Dept. Tech. Sci., power engng. and autom., No. 10. 1960.
51. PETROV, B. N. The Invariance Principle and Its Applicability to Calculation of Linear and Nonlinear Systems. — Proc. First IFAC Congr., Vol. 1. 1960.
52. PIVEN', V. D. The Invariance Problem in Automatic Control Technique. — Proc. First IFAC Congr. 1960.
53. PONTRYAGIN, L. S. et al. Matematicheskaya teoriya optimal'nykh protsessov (Mathematical Theory of Optimal Processes). — Fizmatgiz. 1960.
54. POPOVSKII, A. M. Ob odnom sluchae resheniya zadachi I. N. Voznesenskogo dlya sistem regulirovaniya vzaimosvyazannykh velichin vysokogo poryadka (A Certain Case of the Solution of I. N. Voznesenskii's Problem for Control Systems with High-Order Coupled Variables). — Trudy 2-go Vsesoyuznogo soveshchaniya po teorii avtomaticheskogo regulirovaniya, Vol. 1. 1955.
55. POPOVSKII, A. M. O svobode vybora parametrov avtonomnykh protsessov regulirovaniya neskol'kikh vzaimosvyazannykh velichin (Freedom of Choice of the Parameters of Noninteracting Control Processes with Several Coupled Variables). — Avtomatika i Telemekhanika, 10, No. 6. 1944.

56. POSPELOV, G. S. Invariantnost' nekotorykh koordinat pri avtomaticheskom upravlenii samoletom (Invariance of Some Autopilot Coordinates). — In: "Teoriya invariantnosi i ee primenenie v avtomaticheskikh ustroistvakh", Izdatel'stvo AN SSSR. 1959.
57. ROZONOER, L. I. Nekotorye voprosy teorii svyaznogo regulirovaniya (Some Problems of Interacting Control Theory). — Sbornik Trudov Moskovskogo Energeticheskogo Instituta. 1955.
58. ROZONOER, L. I. Invariantnost' kak variatsionnaya problema (Invariance as a Variational Problem). — Avtomatika i Telemekhanika, No. 5. 1963.
59. SOLOVEI, V. E. Regulirovanie parovoi turbiny s dvumya otboramI para (Controlling a Steam Turbine with Two Steam Outlets). — Avtomatika i Telemekhanika, No. 2. 1939.
60. SOLOV'EV, V. V. Otsenka kachestva svyazannykh sledyashchikh sistem po pokazatelyu kolebatel'nosti (Assessing the Quality of Coupled Servosystems from the Oscillatory Index). — In: "Sbornik nauchnykh trudov Gidroproekta". 1961.
61. SUD-ZLOCHEVSKII, A. M. K voprosu o svyaznom regulirovanii neskol'kikh parametrov (On Interacting Control of Several Parameters). — Avtomatika, No. 3. 1959.
62. SYUZIMOVA, E. M. Nekotorye voprosy svyaznogo regulirovaniya (Some Aspects of Interacting Control). — Avtomatika i Telemekhanika, No. 6. 1960.
63. TEODORCHIK, K. F. and G. A. BENDRIKOV. Analiz mnogokonturnykh lineinykh sistem metodom traektorii kornei (Analysis of Multiloop Linear Systems by the Root Locus Method). — *Izvestiya AN SSSR, Dept. Tech. Sci., power engng. and autom.*, No. 5. 1962.
64. TREGUBOV, A. M. Teoriya peregonki i rektifikatsii (Distillation and Rectification Theory). — Gostekhizdat. 1946.
65. FEL'DBAUM, A. A. *Prosteishie releinye sistemy avtomaticheskogo regulirovaniya* (Simplest Relay Control Systems). — Avtomatika i Telemekhanika, No. 4. 1949.
66. FEL'DBAUM, A. A. Vychislitel'nye ustroistva v avtomaticheskikh sistemakh (*Computers in Automatic Systems*). — Fizmatgiz. 1959.
67. FIALKO, G. M. Primenenie elektrotermicheskikh izodromnykh proizvodstvennykh protsessov (Application of Electrothermal Proportional-plus-Floating Production Processes). — Avtomatika i Telemekhanika, No. 6. 1960.
68. FIALKO, G. M. and D. I. MIL'MAN. Raschet parametrov regulyatorov kontsentratsii gaza po eksperimental'noi snyatoi kharakteristike ob"ektov regulirovaniya (Calculation of the Parameters of Gas Concentration Controllers from an Experimental Characteristic of the Controlled Objects). — Trudy Ural'skogo industrial'nogo instituta. 1954.
69. TSUKERNIK, L. V. Differentsial'nye uravneniya vozmushchennogo dvizheniya slozhnoi energosistemy dlya analiza ee statisticheskoi ustoichivosti (Differential Perturbation Equations of a Complex Power System for the Analysis of Its Statistical Stability). — Izvestiya AN SSSR, Dept. Tech. Sci., power engng. and autom., No. 3. 1956.

70. TSUKERNIK, L. V. and N. A. KOGANOVA. Analiz ustoichivykh slozhnykh energosistem pri pomoshchi elektronnykh schetnykh mashin (Analysis of Stable Complex Power Systems Using a Computer). — Trudy Instituta Elektrotekhniki AN UkrSSR. 1963.
71. TSYPKIN, Ya. Z. Teoriya releinykh sistem (Theory of Relay Systems). — Gostekhizdat. 1955.
72. CHARNYI, I. A. Podzemnaya gazogidrodinamika (Underground Gas and Hydrodynamics). — Gostekhizdat. 1963.
73. CHATTERJEE, H. K. Multivariable Control Systems. — Proc. First IFAC Congr., Vol. 1. 1960.
74. CHINAEV, P. I. O printsipakh sinteza avtomaticheskikh sistem so mnogimi reguliruemymi velichinami (Principles of Synthesis of Multivariable Control Systems). — Avtomatika i Telemekhanika, No. 6. 1960.
75. SHCHELKACHEV, V. N. Razrabotka neftevodonosnykh plastov pri uprugom rezhime (Exploitation of Oil and Water Bearing Strata under Elastic Conditions). — Fizmatgiz. 1960.
76. SHCHIPANOV, G. V. Teoriya i metody postroeniya regulyatorov (Theory and Methods of Controller Design). — Avtomatika i Telemekhanika, No. 1. 1939.
77. BOKSENBOM, A. S. and R. HOOD. General Algebraic Method Applied to Control Analysis of Complex Engine Types. — NACA Tech. Rept. 980, Apr. 1949.
78. FREEMAN, H. A Synthesis Method for Multipole Control Systems. — Trans. AIEE, II: 76, p. II: 28—31. 1957.
79. FREEMAN, H. Stability and Physical Realizability Considerations in the Synthesis of Multipole Control Systems. — Trans. AIEE, 77, p. II: 1—5. 1958.
80. GOLOMB, M. and E. USDIN. A theory of Multidimensional Servosystems. — J. Franklin Inst., No. 1. 1952.
81. KAVANAGH, R. J. The Application of Matrix Methods to Multivariable Control Systems. — J. Franklin Inst., November. 1956.
82. KAVANAGH, R. J. Multivariable Control System Synthesis. — Trans. AIEE, 77, p. II: 425—429. 1958.
83. LUFFER, D. E. and M. L. LOHUSON. Automatic Control of Distillation Columns to Achieve Optimum Operation. — AIChE— IFAC. 1963.
84. MOCZEK, I. S., R. E. OTTO, and T. S. WILLIAMS. Approximation Models to the Dynamic Response of Large Distillation Columns. — AIChE—IFAC. 1963.
85. MESAROVIĆ, M. D. The Control of Multivariable Systems. — New York—London, John Wiley. 1960.
86. TOMOVIĆ, R. Sensitivity Analysis of Dynamic Systems. — Nolit Publishing House, Beograd. 1963.

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Printed in Jerusalem, Israel